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High Fidelity Frequency Response Surface Approximations for Vibration Based Elastic Constants Identification

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Abstract
Some applications such as identification or Monte Carlo based uncertainty quantification often require simple analytical formulas that are fast to evaluate. Approximate closed-form solutions for the natural frequencies of free orthotropic plates have been developed and have a wide range of applicability, but they are not very accurate. For moderate ranges of plate parameters, such as those needed for identifying material properties from vibration tests, good accuracy can be achieved by using response surface methodology combined with dimensional analysis. This paper first demonstrates that such a response surface can be much more accurate then the approximate analytical solutions even for relatively large ranges of the material and geometric parameters. Second it compares the accuracy of the elastic constants identified from experiments using the two approximations. It demonstrates the advantage of high fidelity approximations in vibration-based elastic constants identification. For a least squares identification approach, the approximate analytical solution led to physically implausible properties, while the high-fidelity response surface approximation obtained reasonable estimates.

Keywords: response surface approximation, nondimensionalization, natural frequencies, composites, identification

1. Introduction

Plate vibration has been frequently used for identifying the elastic material parameters of a plate [1], especially composite laminates. The identification is usually done with free-hanging plates in order to avoid difficult-to-model boundary conditions. Bayesian statistical identification approaches [2] have the advantage of handling different sources of uncertainty in the identification procedure. They can also provide confidence intervals and correlation information on the identified properties. However, the Bayesian method can require Monte Carlo simulation which implies large number of vibration calculations for many combinations of the uncertain parameters such as geometry, material parameters, and measurement errors. Numerical solutions (such as finite elements) for free plate vibration natural frequencies are too slow to be used in such a context. Accordingly there is a need for simple approximate analytical formulas that can be evaluated very quickly.

A simple, closed-form approximate analytical solution for the vibration problem of orthotropic plates with free boundary conditions was proposed by Dickinson [3]. This solution is applicable to wide ranges of geometries and materials, but its accuracy might not be sufficient for identification purposes. The aim of the present paper is twofold. First we seek to develop a procedure for a high fidelity, analytical, approximate formula for the natural frequencies of free orthotropic plates based on response surface (RS) methodology. To achieve the desired fidelity the response surface method is combined with dimensional analysis. Our second goal is to compare the effect of the approximation fidelity on the identification results. For this purpose we compare the results of a least squares identification approach using the high fidelity RS approximation and the low fidelity closed-form frequency approximations. This is demonstrated by using experimental data obtained by Pedersen and Fredriksen [4].

In Section 2 we give a quick overview of the approximate analytical solution developed by Dickinson. In Section 3 we apply dimensional analysis to determine the variables of the response surface approximations (RSA) that lead to the best accuracy. In Section 4 we construct the design of experiment for the RSAs and finally in Section 5 we compare their fidelity to finite element analyses and to that of the analytical solution by Dickinson. In Section 6 we present the identification results using the high fidelity approximations and using the low fidelity approximate analytical solution. We provide concluding remarks in Section 7.

2. Existing analytical frequency approximations

The only simple approximate analytical formula for free vibration of orthotropic plates the authors could find was by Dickinson [3], who applied characteristic beam functions in Rayleigh's method to obtain an approximate formula for the flexural vibration of specially orthotropic plates. The formula for free boundary conditions on all four edges is restated below for convenience.

$$f_{mn} = \frac{\pi}{2\sqrt{\rho h}} \sqrt{D_{11}\left(\frac{G_m}{a}\right)^4 + 2H_m H_n D_{12}\left(\frac{1}{a}\right)^2\left(\frac{1}{b}\right)^2 + 4J_m J_n D_{66}\left(\frac{1}{a}\right)^2\left(\frac{1}{b}\right)^2 + D_{22}\left(\frac{G_n}{b}\right)^4} \qquad (1)$$

where f_{mn} is the natural frequency of the mode with wave numbers m and n; ρ is the density of the plate; a, b, h its length, width respectively thickness and D_{ij} the plate flexural rigidities (for detailed expressions of the D_{ij} refer to [5]). G_i, H_i, J_i are constants, depending only on the mode numbers m and n, whose expressions are given in Table 1.

Note that determining the wave number of an experimentally or numerically obtained mode is not always straightforward. In general, the mode number m, respectively n, can be obtained by adding one to the number of nodal lines perpendicular to the edge x respectively y (i.e. it is the number of half wave length in each direction). There are however exceptions, notably for low mode numbers. For a detailed study of the modes and associated mode numbers of free plates refer to [6].

Table 1: Expression of the coefficients in the approximate formula for natural frequencies of Eq. 1.

Mode index i	G_i	H_i	J_i
1	0	0	0
2	0	0	1.216
3	1.506	1.248	5.017
i (i>3)	$i-\frac{3}{2}$	$\left(i-\frac{3}{2}\right)^2\left(1-\frac{2}{(i-3/2)\pi}\right)$	$\left(i-\frac{3}{2}\right)^2\left(1+\frac{6}{(i-3/2)\pi}\right)$

Dickinson's simple analytical expression is computationally inexpensive, thus a priori suitable for use in statistical methods which require its repeated use a large number of times. However the fidelity of the approximation must also be acceptable for such a use. Typically this analytical approximation was reported to be within 5% of the exact numerical solution [7]. It is not clear whether this accuracy is sufficient when used for identifying accurate elastic constants from vibration experiments. Therefore, in the next sections we will also develop more accurate response surface approximations of the natural frequencies.

3. Determining nondimensional variables for the RSA

Response surface methodology, also called surrogate modeling, is a technique used to approximate the response of a process which is known only in a finite and usually small number of points. The points where the response is known, which constitute the design of experiments (DoE), are fitted with a particular function depending on the RSA type used. A common RS type is the polynomial response surface (PRS), which uses least-square fit to obtain a polynomial approximation. For more details on RSA techniques refer to [8].

For elastic constants identification, we propose to fit to finite element simulations, PRS of the natural frequencies of the plate in terms of the elastic constants and parameters that may have some uncertainty in their values : ρ, a, b, h and the four D_{ij}.. We could directly construct a polynomial response surface as a function of these individual model parameters. However, the accuracy of the RSA is generally improved and the number of required simulations is reduced if the number of variables is reduced by using the nondimensional parameters characterizing the problem [9]. To find these parameters we nondimensionalize the equations describing the vibration of a symmetric, specially orthotropic laminate.

Governing equation: $D_{11}\frac{\partial^4 w}{\partial x^4}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}+\rho h\frac{\partial^2 w}{\partial t^2}=0$

where w is the out of plane displacement.

Boundary conditions:
On edge $x = 0$ and $x = a$ (denoted $x = 0;a$):

$$M_x=0 \quad \Leftrightarrow \quad -D_{11}\frac{\partial^2 w}{\partial x^2}\bigg|_{x=0;a}-D_{12}\frac{\partial^2 w}{\partial y^2}\bigg|_{x=0;a}=0$$

$$Q_x+\frac{\partial M_{xy}}{\partial y}=0 \quad \Leftrightarrow \quad -D_{11}\frac{\partial^3 w}{\partial x^3}\bigg|_{x=0;a}-\left(D_{12}+4D_{66}\right)\frac{\partial^3 w}{\partial x\partial y^2}\bigg|_{x=0;a}=0$$

On edge $y = 0$ and $y = b$ (denoted $y = 0;b$):

$$M_y=0 \quad \Leftrightarrow \quad -D_{12}\frac{\partial^2 w}{\partial x^2}\bigg|_{y=0;b}-D_{22}\frac{\partial^2 w}{\partial y^2}\bigg|_{y=0;b}=0$$

$$Q_y+\frac{\partial M_{xy}}{\partial x}=0 \quad \Leftrightarrow \quad -D_{22}\frac{\partial^3 w}{\partial y^3}\bigg|_{y=0;b}-\left(D_{12}+4D_{66}\right)\frac{\partial^3 w}{\partial x^2\partial y}\bigg|_{y=0;b}=0$$

This vibration problem involves 11 variables to which we add the variable of the natural frequencies f_{mn} that we seek, so a total of 12 variables for the problem of determining the plate's natural frequency (see Table 2).

Table 2: Variables involved in the vibration problem and their units

Variable	f_{mn}	w	x	y	a	b	t	ρh	D_{11}	D_{12}	D_{22}	D_{66}
Unit	$\frac{1}{s}$	m	m	m	m	m	s	$\frac{kg}{m^2}$	$\frac{kg\cdot m^2}{s^2}$	$\frac{kg\cdot m^2}{s^2}$	$\frac{kg\cdot m^2}{s^2}$	$\frac{kg\cdot m^2}{s^2}$

These 12 variables involve 3 dimension groups (m, kg, s). According to the Vaschy-Buckingham theorem [10][11] we know that we can have a minimum of 12 – 3 = 9 nondimensional groups.

Posing $\tau=\sqrt{\frac{\rho h a^4}{D_{11}}}$, which is a characteristic time constant, the 9 nondimensional groups are given in Table 3:

Table 3: Nondimensional parameters characterizing the vibration problem

Nondimensional Parameters	$\Omega=\frac{w}{h}$	$\theta=\frac{t}{\tau}$	$\xi=\frac{x}{a}$	$\eta=\frac{y}{b}$
$\Psi_{mn}=\tau f_{mn}$	$\Delta_{12}=\frac{D_{12}}{D_{11}}$	$\Delta_{22}=\frac{D_{22}}{D_{11}}$	$\Delta_{66}=\frac{D_{66}}{D_{11}}$	$\gamma=\frac{a}{b}$

As function of these nondimensional variables the vibration problem can be written as follows:

Governing equation: $\frac{\partial^4\Omega}{\partial\xi^4}+2\left(\Delta_{12}+2\Delta_{66}\right)\gamma^2\frac{\partial^4\Omega}{\partial\xi^2\partial\eta^2}+\Delta_{22}\gamma^4\frac{\partial^4\Omega}{\partial\eta^4}+\frac{\partial^2\Omega}{\partial\theta^2}=0$

Boundary conditions:
On edge $\xi = 0$ and $\xi = 1$ (denoted $\xi = 0;1$):

$$-\frac{\partial^2\Omega}{\partial\xi^2}\bigg|_{\xi=0;1}-\Delta_{12}\gamma^2\frac{\partial^2\Omega}{\partial\eta^2}\bigg|_{\xi=0;1}=0$$

$$-\frac{\partial^3\Omega}{\partial\xi^3}\bigg|_{\xi=0;1}-\left(\Delta_{12}+4\Delta_{66}\right)\gamma^2\frac{\partial^3\Omega}{\partial\xi\partial\eta^2}\bigg|_{\xi=0;1}=0$$

On edge η = 0 and η = 1 (denoted η = 0;1):

$$-\Delta_{12}\left.\frac{\partial^2\Omega}{\partial\xi^2}\right|_{\eta=0;1}-\Delta_{22}\gamma^2\left.\frac{\partial^2\Omega}{\partial\eta^2}\right|_{\eta=0;1}=0$$

$$-\Delta_{22}\gamma^3\left.\frac{\partial^3\Omega}{\partial\eta^3}\right|_{\eta=0;1}-\left(\Delta_{12}+4\Delta_{66}\right)\gamma\left.\frac{\partial^3\Omega}{\partial\xi^2\partial\eta}\right|_{\eta=0;1}=0$$

For finding an RSA of the nondimensional natural frequency Ψ_{mn}, we are not interested in the vibration mode shapes and we do not need the nondimensional out-of-plane displacement Ω, nor the nondimensional time θ, nor the nondimensional coordinates ξ and η. This means that the nondimensional natural frequency Ψ_{mn} can be expressed as a function of only four nondimensional parameters $\Psi_{mn} = \Psi_{mn}(\Delta_{12}, \Delta_{22}, \Delta_{66}, \gamma)$.

Note that rewriting the analytical approximation of Eq. 1 in its nondimensional form leads to a polynomial function of the nondimensional parameters:

$$\left(\psi_{mn}\right)^2=\frac{\pi^2}{4}\left(G_m^4+2H_mH_n\Delta_{12}\gamma^2+4J_mJ_n\Delta_{66}\gamma^2+\Delta_{22}\gamma^4G_n^4\right) \tag{2}$$

Equation 2 is a cubic polynomial in Δ_{12} , Δ_{22} , Δ_{66} and γ^2 . We therefore expressed the squared nondimensional frequency as a cubic polynomial response surface (PRS) in terms of these four variables. Such a PRS has 31 additional polynomial terms beyond those in those in Eq. 2, that can potentially increase the fidelity of the response surface approximation.

4. Constructing the RSA

The RSA will be fitted to finite element (FE) simulations of the plate using Abaqus® commercial FE software. We used 400 thin plate elements (S8R5) to model the composite plate.

To fit the RSA we need to sample points in the four-dimensional space of the nondimensional parameters. The ranges depend on the application, and we selected experiments carried out by Pedersen and Fredriksen [4] for comparing the analytical and RS approximations. Had we sampled in the nondimensional variables directly, it would have been difficult to deduce values for the dimensional variables needed for the FE model (E_1, E_2, ν_{12}, G_{12}, a, b, h and ρ). Accordingly we chose the following procedure to obtain the points in the nondimensional space and their corresponding dimensional parameters:

i. Sample 5000 points in the eight dimensional-variables space { E_1, E_2, ν_{12}, G_{12}, a, b, h, ρ } with uniform Latin Hypercube sampling within the bounds considered for the problem.

ii. Out of the 5000 points extract 200 points in the nondimensional space by maximizing the minimum (max-min) distance between any two points. These steps ensure that the points are well distributed (space-filling) in the nondimensional space.

Figure 1 illustrates this procedure in a two-dimensional case with Δ_{12} and γ only. The blue crosses are representative of the 5000 points sampled in step i. The red circles are representative of the 200 points selected in step two. Because we stopped the max-min search after 100,000 iterations (to keep computational cost reasonable) we did not get perfectly distributed points, but this is not required for good accuracy of the RSA.

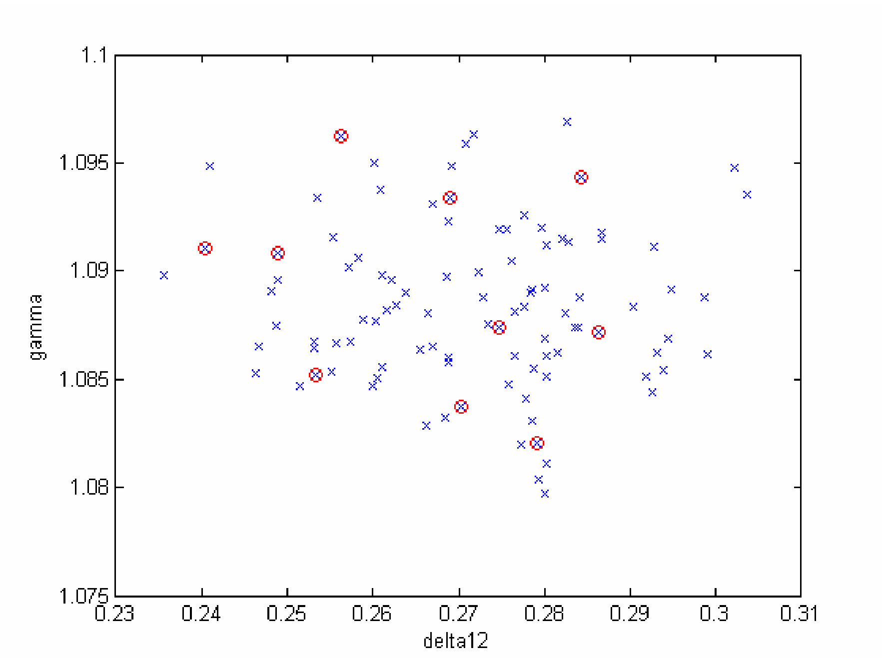

Figure 1: Illustration of the procedure for sampling points in the nondimensional space.

5. Frequency RSA results

The bounds on the variables given in Table 4 were chosen based on the vibration based identification problem described in [4]. In our case the RSA would be used to carry out least squares and later Bayesian identification approaches for the material properties. The plate is a glass-epoxy composite panel with stacking sequence $[0,-40,40,90,40,0,90,-40]_s$. We decided to construct two sets of RSAs with two different bounds. Indeed, based on trial and error we found that the RSA for least squares based identification required to have wider bounds than for Bayesian identification or the identified values would fall outside of the RSA construction bounds. Narrow bounds would also be useful for a refined least squares identification over a small domain around the optimum obtained with the larger bounds. Table 4 presents the wide bounds (denoted WB) used for constructing the first RSA, that will be used for least squares based identification in Section 6.

Table 4: Wide Bounds on the model input parameters (denoted WB)

	E_1 (GPa)	E_2 (GPa)	ν_{12}	G_{12} (GPa)	a (mm)	b (mm)	h (mm)	ρ (kg/m^3)
Low bound	43	15	0.2	7	188	172	2.2	1800
High bound	80	28	0.36	13	230	211	3.0	2450

We constructed a cubic polynomial response surface approximations (PRS) for each of the first ten squared nondimensional natural frequencies as a function of the nondimensional parameters determined in Section 3. We used the procedure described in the previous section but sampling 250 points instead of 200 within the bounds WB. The response surface approximations fitted through the 250 points are denoted RSA_{WB}.

To test the accuracy of the RSAs we tested them at an additional 250 finite element points (denoted P250), sampled using the same procedure as described in the previous section, using the bounds given in Table 4. The results are given in Table 5. The reader can also refer to Table 9 to get an idea of the order of magnitude of the different frequencies.

Table 5: Mean and maximum relative absolute error of the nondimensional frequency RSA predictions (denoted RSA_{WB}) compared at 250 test points

Absolute Error (%)	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
mean	0.0327	0.5476	0.2897	0.0318	0.0381	0.6802	0.667	0.5834	1.1097	0.5899
max	0.1745	4.1972	1.6954	0.1399	0.1951	5.2187	5.6103	3.6798	7.8341	7.4978

For comparison purpose we also provide in Table 6 the error of the analytical frequency approximation of Eq. 1

compared at these same 250 test points.

Table 6: Mean and maximum relative absolute error of the analytical formula frequency predictions compared at 250 test points

Absolute Error (%)	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
mean	5.98	8.50	3.47	4.28	7.22	4.68	2.77	5.72	5.75	1.22
max	6.54	16.28	8.06	9.43	23.32	21.11	18.30	10.28	12.05	10.01

The average error in the analytical approximation over the first ten frequencies was found to be 4.9%. This is consistent with previous studies [7] which reported the error of using the analytical formula to be about 5%. On the other hand the errors in the RSAs are about an order of magnitude lower.

The second RSA set we construct is for the narrower bounds given in Table 7. These would be typical for a Bayesian identification considering typical uncertainty ranges. These bounds could also be used in least squares identification for refining the optimum.

Table 7: Narrow bounds on the model input parameters (denoted NB)

	E_1 (GPa)	E_2 (GPa)	ν_{12}	G_{12} (GPa)	a (mm)	b (mm)	h (mm)	ρ (kg/m^3)
Low bound	52	18	0.23	8.3	202	185	2.55	2000
High bound	70	25	0.32	11	216	200	2.65	2240

Cubic PRS were again fitted for each of the first ten nondimensional natural frequencies using 200 finite element simulations. The RSA fidelity was tested at 250 additional points sampled within the bounds of Table 7. The results are presented in Table 8.

Table 8: Mean and maximum relative absolute error of the nondimensional frequency RSA predictions (denoted RSA_{NB}) compared at 250 test points

Absolute Error (%)	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
mean	0.0043	0.0036	0.0028	0.0045	0.0044	0.0033	0.0029	0.0049	0.0062	0.0046
max	0.0212	0.0328	0.0301	0.016	0.0197	0.0155	0.0096	0.0171	0.0613	0.0413

For these smaller bounds the RSA fidelity achieved was excellent, the mean of the error among the 250 test points being smaller than 0.01% for all the frequencies. The maximum error among the 250 test points was found to be only about 0.06% for the 9th frequency.

6 Application to vibration based identification

6.1 Identification scheme

The frequency approximations were constructed to be used with a Bayesian identification procedure of elastic constants. The Bayesian procedure allows to account for uncertainties on other input parameters (plate dimensions, density) when identifying the material properties. This explains why we wanted the RSA to account for variations in these parameters. Nondimensionalization allows in this case to condense the different variables in the reduced number of nondimensional parameters. If we were interested only in variations in the elastic constants this would have been of more limited interest.

Before looking at the Bayesian problem in a future study, we investigate in the present section the effect of approximation fidelity on the more simple problem of least squares identification of the elastic constants. In this case we only have four variables (the elastic constants) but it already allows to compare the identifications with the low fidelity analytical approximate solution and with the higher fidelity RSA.

The identification procedure seeks the four orthotropic ply elastic constants (E_1, E_2, ν_{12}, and G_{12}) of a glass/epoxy composite based on the first ten natural frequencies of a $[0,-40,40,90,40,0,90,-40]_s$ laminate vibrating under free boundary conditions. We use the values measured by Pedersen and Frederiksen [4] as experimental frequencies in the identification procedure. For convenience these measured frequencies are also given in Table 9 and the plate properties and dimensions in Table 10.

Table 9 : Experimental frequencies from Pedersen and Frederiksen [4]

Frequency	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
Value (Hz)	172.5	250.2	300.6	437.9	443.6	760.3	766.2	797.4	872.6	963.4

Table 10: Plate properties: length (a), width (b), thickness(h) and density (ρ)

Parameter	a (mm)	b (mm)	h (mm)	ρ (kg/m^3)
Value	209	192	2.59	2120

The identification scheme is a basic least squares approach. The identified parameters correspond to the minimum of the following objective function:

$$J(E)=\sum_{i=1}^{m}\left(\frac{f_i^{num}(E)-f_i^{measure}}{f_i^{measure}}\right)^2 \quad (3)$$

where $E = \{ E_1, E_2, \nu_{12}, G_{12}\}$, $f_i^{measure}$ is the ith experimental frequency from Table 9 and f_i^{num} is a numerical frequency prediction (finite elements here).

6.2 Identification using the response surface approximation

As mentioned earlier the least squares identification will use the RSAs with large bounds (denoted RSA_{WB}, see Section 5). The least squares (LS) optimization was carried out without imposing any bounds on the variables during the optimization. This led to the optimum shown in Table 13. Note that in [4] Pedersen and Frederiksen applied a least squares approach coupled directly to a numerical model (Rayleigh-Ritz) to identify the elastic constants. The properties that they found are considered here as reference values and are also provided in Table 13.

Table 13: LS identified properties using the frequency RSAs

Parameter	E_1(GPa)	E_2 (GPa)	ν_{12}	G_{12} (GPa)
Identified values	60.9	22.7	0.217	9.6
Reference values [4]	61.3	21.4	0.279	9.8

The residuals between the frequencies at the optimal points and the experimental frequencies are given in Table 14. They are relatively small and the identified values are also reasonably close to the reference values. This means that the accuracy of the RSA_{WB} is good enough to lead to reasonable results. This is not surprising since the RSAs have good accuracy allowing the least squares optimization to unfold properly.

Table 14: Residuals for LS identification using the frequency RSAs. $J(E) = 1.7807\ 10^{-4}$.

Frequency	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
Residual (%)	0.11	-0.60	0.09	0.88	-0.25	0.30	-0.18	0.38	-0.46	-0.30

In the next section we investigate the identification results obtained with the lower fidelity analytical approximate solution [3] which has much poorer accuracy. This could lead to more significant differences between the two identification methods.

6.3 Identification using the analytical approximate solution

Using the analytical approximate solution, the least squares (LS) optimization was carried out first while imposing bounds on the variables during the optimization. We imposed on E_1, E_2, ν_{12}, and G_{12} the bounds given in Table 7, which seem reasonable for the properties that we are seeking. The results of the optimization are given in Tables 16 and 17. The norm of the residuals (i.e. the value of the objective function) is $J(E) = 0.019812$.

Table 16: LS identified properties using the analytical approximate solution (bounded variables)

Parameter	E_1(GPa)	E_2 (GPa)	ν_{12}	G_{12} (GPa)
Identified values	52.0	25.0	0.298	8.3
Reference values [4]	61.3	21.4	0.279	9.8

Table 17: Residuals for LS identification using the analytical approximate solution. $J(E)$ = 0.019812.

Frequency	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
Residual (%)	-1.27	7.06	-6.84	1.64	-1.37	-2.91	0.72	1.77	6.19	-6.70

We note that several variables hit the bounds. We could keep these results since the bounds we imposed are quite wide and from a physical point of view it is quite unlikely that the true parameters lie outside the bounds. We wanted however to also know what happens when imposing no bounds at all.

The unbounded optimum found is given in Table 18. The residuals between the frequencies at the optimal points and the experimental frequencies are given in Table 19. The norm of the residuals (i.e. the value of the objective function) is $J(E)$ = 0.019709.

Table 18: LS identified properties using the analytical approximate solution (unbounded variables)

Parameter	E_1(GPa)	E_2 (GPa)	ν_{12}	G_{12} (GPa)
Identified values	71.1	46.2	-0.402	-17.1
Reference values [4]	61.3	21.4	0.279	9.8

Table 19: Residuals for LS identification using the analytical approximate solution. $J(E)$ = 0.019709.

Frequency	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
Residual (%)	-1.24	6.96	-6.77	1.69	-1.38	-3.00	0.79	1.48	6.25	-6.75

It is obvious from the identified results that the unbounded optimum found is not plausible. Not only are the parameters very far away from the reference values but the Poisson's ratio and shear modulus have negative values. While a negative Poisson's ratio is physically possible, negative shear modulus has no physical meaning.

It is also worth noting that in spite of the implausible optimum the residuals, whether for the bounded or unbounded case, are not very large. All are of the order of a few percent, which for recall is also the order of the accuracy of the analytical approximate solution compared to finite element analyses (see Table 6). We can also note that the residuals and their norm remain practically unchanged for the unbounded optimization compared to the bounded one (Table 17). This is a sign of the ill-conditioning of the least squares problem due to a very flat objective function around the optimum. It hints that the accuracy of the frequency approximation has a large effect on the identified results and while a few percent error might seem very reasonable for some application, it can lead to extremely bad results when applied to the present identification problem.

Independently whether the bounded or unbounded optimum, the least squares identification results using the analytical approximate solution from [3] are significantly further away from the reference values than the results with the high fidelity RSA. This illustrates the importance of high fidelity frequency approximations for the present vibration based identification problem.

7. Conclusion

In a first part a procedure was detailed allowing to obtain simple polynomial response surface approximations (RSA) for the natural frequencies of a vibrating orthotropic plate. While the procedure and the obtained expressions are relatively simple, it can achieve high fidelity, allowing it to be used in most applications that require fast function evaluations together with high fidelity such as Monte Carlo simulation for Bayesian identification analysis. The RSAs constructed were between one and two orders of magnitudes more accurate than previously existing approximate analytical formulas for vibration of free orthotropic plates. To achieve such high fidelity the RSAs were fitted to the nondimensional parameters characterising the vibration problem.

Note that the overall procedure is applicable not only to free but any boundary conditions as long as the RSAs are refitted to the corresponding design of experiments in terms of the nondimensional parameters characterizing the vibration problem with the specific boundary conditions.

In the latter part we showed that the fidelity of the frequency approximation has significant impact on the material properties identification problem we consider. The high fidelity nondimensional frequency RSAs led to reasonable results. On the other hand low fidelity frequency approximations such as the analytical approximate solution in [3], that had an accuracy of about 5%, led to unreasonable or even physically implausible identification results.

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