

How adaptively constructed reduced order models can benefit sampling based methods for reliability analyses

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Abstract. Many sampling based approaches are currently available for calculating the reliability of a design. The most efficient methods can achieve reductions in the computational cost by one to several orders of magnitude compared to the basic Monte Carlo method. This paper is specifically targeted at sampling based approaches for reliability analysis, in which the samples represent calls to expensive finite element models. The aim of this paper is to illustrate how these methods can further benefit from reduced order modeling to achieve drastic additional computational cost reductions, in cases where the reliability analysis is carried out on finite element models. Standard Monte Carlo, importance sampling, separable Monte Carlo and a combined importance separable Monte Carlo approach are presented and coupled with reduced order modeling. An adaptive construction of the reduced basis models is proposed. The various approaches are compared on a thermal reliability design problem, where the coupling with the adaptively constructed reduced order models is shown to further increase the computational efficiency by up to a factor of six.

Keywords: reliability analysis, Monte Carlo simulation, reduced order models, reduced basis, on the fly construction

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Nomenclature

α – reduced state variables

β – reliability index

ϕ – matrix formed by the reduced basis vectors

$\Phi()$ - cumulative distribution function of the standard normal distribution

μ – parameters of interest of the finite element model

C – capacity used in the limit state function G

COV – coefficient of variation

e_{RB} – normalized residual

F – vector of the loadings

$G()$ – limit state function in reliability analysis

h – film convection coefficient

i – number of iteration

$I()$ - indicator function, equal to 1 if the condition is met and equal to 0

k – thermal conductivity

K – matrix corresponding to the matrix form of the finite element equilibrium equations

M – number of samples of the capacity in separable Monte Carlo

N – number of samples in Monte Carlo and importance sampling

P_f – probability of failure

R – response used in the limit state function G

T – temperature

U - standard normal mapping of the uncertain parameters X

V – vector of the unknown state variable on which the response R of the limit state function depends

V_{RB} – vector of the reduced basis approximation of the state variable

X – uncertain input parameters in reliability analysis

1 Introduction

Reliability analysis is receiving growing interest from industry as a tool for increasing competitiveness by allowing to design closer to the design limits. This has spurred a large number of methodological developments. Analytical approaches to reliability estimation [1],[2] are fast to evaluate but are not always possible on industry problems. Sampling based approaches (i.e. Monte Carlo simulation) are very popular approaches due to their implementation simplicity. The basic Monte Carlo approach suffers however from high computational cost when low probabilities of failure are being sought. This is due to the large number of simulations that need to be carried out in such cases. When each simulation is the output of a finite element model, as can be often the case for industry applications, the overall cost quickly becomes intractable. Several methods have been developed in the past allowing to decrease the number of samples required for a given accuracy on the probability of failure estimate. Such methods, that we refer to as advanced Monte Carlo approaches, include importance sampling [3][4], separable Monte Carlo [5][6], Markov chain Monte Carlo [7][8]. Combinations of these methods have been shown to provide synergies in further decreasing the overall cost [9]. Other reliability estimation methods, including surrogate based approaches [10]-[14] have also been shown to be very efficient at reducing the computational cost compared to the basic Monte Carlo sampling.

Another way to decrease the cost of sampling based approaches is to reduce the computational cost of each simulation. This can be achieved by the use of reduced order models. Reduced order models provide an approximation of the solution of the exact equilibrium equations with drastically reduced computational cost, and can thus partly or fully replace the exact solution during the Monte Carlo method. Such approaches have gained

much interest in recent years in various domains [15]-[20]. Reduced order models were initially developed in structural dynamics [21] and recent developments allowed coupling of reanalysis techniques with several uncertainty propagation methods [22],[23]. Different approaches to model order reduction have proved applicable however not only to vibration problems but to a wide range of problems involving a parametric model of the system as illustrated in the recent survey by Benner et al. [24]. A particular type of reduced order model is the so called reduced order model by projection or reduced basis model, in which the equilibrium equations are being solved projected on a certain basis, which is usually of much lower dimension than the size of the system of equilibrium equations themselves. The low dimensionality allows significant computational savings, typically by more than an order of magnitude. Different approaches have been proposed in recent years to make use of reduced basis models within the basic Monte Carlo framework [25],[26]. These methods have shown the potential of this coupling but the results obtained still suffer from the fact that basic Monte Carlo is quite inefficient at finding low probabilities of failure. Note that many other methods for carrying out reliability analyses of complex systems are available such as Bayesian reliability analysis [27], failure mode and effect analysis [28], failure mode, effects and criticality analysis [29] or allocation-optimization for reliability-redundancy allocation [30].

The present paper is specifically targeted at reliability analyses that make use of calls to expensive finite element models in order to calculate low probabilities of failure using some of the sampling methods reviewed before (Monte Carlo, separable Monte Carlo, Markov chain Monte Carlo, importance sampling). One of the major challenges with such analyses is their computational cost, which can become very significant when large finite element models are used. This challenge can essentially be addressed from two fronts: by developing new reliability analysis methods or by developing more efficient ways to solve expensive finite element problems, for example through the use of reduced order models. In the present paper we build upon existing developments in each of these two domains. The novelty of the paper

resides in proposing an adaptive coupling between reduced order modeling and existing reliability analysis methods. The synergies obtained by the proposed hybrid approach have the potential to lead to significant further computational cost savings of reliability analysis that need to call expensive finite element models.

The aim of this paper is to provide and illustrate an adaptive coupling of a specific type of reduced order model (i.e. a reduced basis model) with some of the advanced Monte Carlo methods previously reviewed and provide a comparative overview of the computational savings potential of the different approaches. For this we will compare basic Monte Carlo, importance sampling, separable Monte Carlo and combined importance sampling with separable Monte Carlo. The choice of these reliability estimation methods is not restrictive and other methods, including surrogate based methods, could potentially also be coupled with reduced basis modeling as will be mentioned in the concluding remarks of the paper. An approach, as well as the corresponding algorithm, for adaptively coupling the construction of the reduced order model with each of the selected methods will be presented and the effect of this coupling on the computational efficiency will be compared for each of them.

The rest of the paper is organized as follows. In section 2 theoretical formulations of various reliability approaches as well as of the reduced basis method are recalled. Section 3 presents the reduced basis strategy for each of the presented reliability formulations: basic Monte Carlo, importance sampling, separable Monte Carlo and the combined importance sampling separable Monte Carlo approach. Section 4 compares the results of the various methods on a heat transfer finite element problem. Finally section 5 provides concluding remarks.

2 Background

2.1 Reliability analysis formulation

Let us consider a limit state function $G(\mathbf{X})$, which is a function of the uncertain input parameters $\mathbf{X}$ of the problem considered. The limit state function is defined such that $G(\mathbf{X}) < 0$ when failure occurs and $G(\mathbf{X}) > 0$ when failure does not occur. The probability of failure P_f can then be defined as shown in Eq.1.

$$P_f = P(G(\mathbf{X}) < 0) \quad (1)$$

The challenge in reliability analysis lies in calculating this probability as efficiently as possible. A basic, yet very efficient way of approximating this probability of failure is by the first order reliability method (FORM) [31]. Its fundamental idea lies in approximating the limit state by a first order approximation at the most probable point of failure. In FORM the vector of the random variables $\mathbf{X}$ is mapped into a standard normal vector $\mathbf{U}$. The transformation from $\mathbf{X}$ to $\mathbf{U}$ is known as the Rosenblatt transformation [32]. Note that the limit state function needs also to be transformed. We denote by G' the limit state function in the standard normal space. In order to find the most probable point (MPP) of failure the following optimization problem is solved:

$$\begin{aligned} \text{find } \mathbf{U}^* \text{ solution of : } \min_{\mathbf{U}} \sqrt{\mathbf{U}^T \mathbf{U}} \\ \text{s.t. : } G'(\mathbf{U}) = 0 \end{aligned} \quad (2)$$

The point $\mathbf{U}^*$ is called the most probable point of failure since it is the closest to the origin in the standard normal space. The distance from this point to the origin is denoted by $\beta = \sqrt{\mathbf{U}^{*T} \mathbf{U}^*}$.

The probability of failure can then be approximated by Eq. 3.

$$P_f^{FORM} = \Phi^{-1}(-\beta) \quad (3)$$

where Φ is the cumulative distribution function of the standard normal distribution.

Note that the FORM method is usually very efficient computationally, however the approximation may be quite poor if the limit state function cannot be accurately approximated by its first order. Accordingly sampling based methods have been developed which do not suffer from this drawback, but at the expense of usually higher computational costs.

2.2 Monte Carlo and Separable Monte Carlo

The probability of failure of Eq. 1 can be expressed in integral form as shown in Eq. 4.

$$P_f = \int I[G(x) < 0] f_X(x) dx \quad (4)$$

Where $I[\bullet]$ is the indicator function, equal to 1 if the condition is met and equal to 0 otherwise. The function f_X is the probability density function of the random variable X .

The Monte Carlo estimation of this probability of failure $\hat{P}_{f_{Standard\ MC}}$ is then given by Eq. 5 based on $N_{Standard\ MC}$ samples of the random variable X .

$$\hat{P}_{f_{Standard\ MC}} = \frac{1}{N_{Standard\ MC}} \sum_{i=1}^N I[G(X_i) < 0] \quad (5)$$

This estimate of the probability of failure depends of course on the samples that are being used [33]. The coefficient of variation, COV (i.e. the standard deviation over the mean) of $\hat{P}_{f_{Standard\ MC}}$ characterizing this variability due to sampling can be expressed by Eq. 6.

$$COV(\hat{P}_{f_{Standard\ MC}}) = \sqrt{\frac{(1 - P_{f_{Standard\ MC}})}{P_{f_{Standard\ MC}} N_{Standard\ MC}}} \quad (6)$$

An approximation of this COV can be easily obtained by considering $P_{f_{Standard\ MC}} = \hat{P}_{f_{Standard\ MC}}$.

While the Monte Carlo estimate of the probability of failure is very simple to implement it has the major drawback of requiring a large number of samples in order to achieve reasonably small coefficients of variation, especially when low probabilities of failure are being sought.

This situation can be easily improved under an assumption of statistical independence. Indeed the limit state function $G(\mathbf{X})$ can be very often expressed as a function of a response R and a capacity C . Typically $G(\mathbf{X}) = C(\mathbf{X}) - R(\mathbf{X})$ such that if the response is higher than the capacity there is failure.

The Separable Monte Carlo (SMC) method [5] is a variation of the Monte Carlo method, which was designed specifically to take advantage of a common situation where the response, R , and capacity, C , are stochastically independent random variables. Given this independence the limit state function can be sampled separately for response and capacity, which has the potential of requiring fewer expensive samples for estimating a probability of failure. We consider that the uncertainty in capacity depends on a set of random variables, $\mathbf{X}_C$ and that in the response depends on a different set of random variables, $\mathbf{X}_R$ which are mutually independent. The limit state for probability of failure calculation can then be expressed as shown in Eq. 7.

$$G(C, R) = G(C(\mathbf{X}_C), R(\mathbf{X}_R)) \quad (7)$$

Given this separable form we can independently draw samples M samples of capacity $\mathbf{X}_C$ and N samples of response $\mathbf{X}_R$ and evaluate all possible combinations of these samples to evaluate when failure does occur. This creates a large sample of points with only modest number of samples of either response or capacity. Figure 1 provides an overview of the difference of how Standard MC and Separable MC treats given samples. While one draw of R is compared only with the corresponding draw of C in Standard MC, in separable MC each draw of R is compared with all the draws of C .

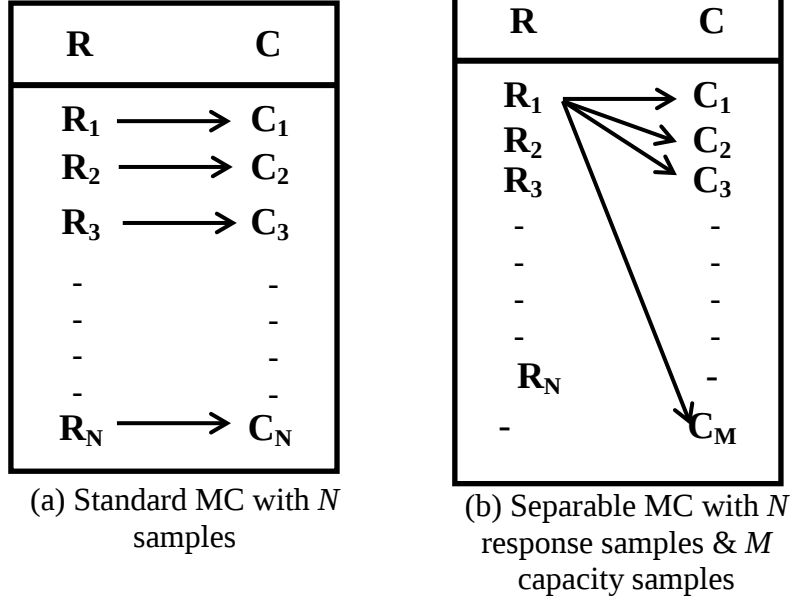


Figure 1. Illustration (from [6]) of the difference between (a) Standard MC and (b) Separable MC

Separable MC allows different sample sizes for response and capacity, which is very advantageous when working with limited computational budget, since a smaller sample size for the computationally expensive calculation (usually the response) can be somewhat compensated by more samples of the computationally cheap calculation (usually the capacity).

The probability of failure estimate, $\hat{P}_{f_{Separable MC}}$, by the Separable Monte Carlo method is provided in Eq. 8.

$$\hat{P}_{f_{Separable MC}} = \frac{1}{MN} \sum_{i=1}^M \sum_{j=1}^N I[G(X_{C_i}, X_{R_j}) < 0] \quad (8)$$

Variance expressions for this estimate are also available but depend on the form of the limit state function and can not be as easily applied as Eq. 6 due to the presence of covariance terms. The interested reader is referred to [5] for more details.

2.3 Importance sampling

Importance sampling [34] is an advanced Monte Carlo sampling procedure based on the use of a new sampling density function determined such as to pick “important” values of the input random variables for the probability calculation under consideration. In the case of probability of failure, the important regions are the regions of relative high probability where the limit state is near zero, because this is the region most involved with failure.

Using such a modified sampling density improves the accuracy of estimation of the statistical response of interest, the probability of failure here. To compensate for the use of different sampling densities, the samples are weighted.

To ensure that most of the sampled points are in the failure region, a new sampling distribution centered in the failure region $h_X(\cdot)$ is selected from which the set of random variables are sampled. The probability of failure based on this new sampling density can be expressed as shown in Eq. 9 both in the initial random variables' space X and in the standard normal variables' space U .

$$\begin{aligned} P_{f_{IS}} &= \int I[G(x) < 0] \frac{f_X(x)}{h_X(x)} h_v(x) dx \\ &= \int I[G'(u) < 0] \frac{\Phi(u)}{h_U(u)} h_U(u) du \end{aligned} \quad (9)$$

The selection of the new sampling density $h(\cdot)$ has to be such that maximum information can be extracted out of the samples generated, which for the probability of failure should be ideally around the highest likelihood of $f_X(\cdot)$ that still lies on the failure surface $G(X)=0$. In this paper, we use a normal distribution centered at the Most Probable Point (MPP) of failure and having a coefficient of variation equal to that of the initial distribution as suggested by Melchers [34].

The importance sampling estimation $\hat{P}_{f_{IS}}$ of the probability of failure can then be obtained as shown in Eq. 10.

$$\hat{P}_{fs} = \frac{1}{N} \sum_{i=1}^N I[G(X_i) < 0] \frac{f_X(X_i)}{h_V(X_i)} \quad (10)$$

where $\frac{f_X(x)}{h_V(x)}$ is called the weight function.

Note that it is advantageous to express this estimator in the standard normal space $\mathbf{U}$ as shown in Eq. 11 [31], such as to simplify the calculation of the weight function.

$$\hat{P}_{fs} = \frac{1}{N} \sum_{i=1}^N I[G'(U_i) < 0] \frac{\Phi(U_i)}{h_U(U_i)} = \frac{1}{N} \sum_{i=1}^N I[G'(U_i) < 0] \exp\left(-U_i^T U^* - \frac{\beta^2}{2}\right) \quad (11)$$

where superscript T denotes the transpose and β is the distance from the origin to the MPP point in the standard normal space and U^* are the coordinates in the standard normal space of the MPP point.

An appealing way to further reduce the computational burden related to importance sampling is to combine it with the previously described separable Monte Carlo approach when the response and capacity are independent. The sampling for M samples of capacity and N samples of response is done using the importance sampling strategy of sampling around the most probable point of failure. Then, as in Separable MC, all combinations of M capacity samples and N response samples are compared for evaluating failure. This creates a large sample of points using a modest number of samples for either response or capacity, which compared to the basic separable Monte Carlo approach of section 2.2 are mostly around the failure region. The Monte Carlo estimation for probability of failure using importance sampling based separable Monte Carlo, ImpSMC [9] is provided in Eq. 12.

$$\hat{P}_{f_{\text{ImpSMC}}} = \frac{1}{MN} \sum_{i=1}^M \sum_{j=1}^N I[G'(U_{C_i}, U_{R_j}) < 0] \exp\left(-\begin{pmatrix} U_{R_j} \\ U_{C_i} \end{pmatrix}^T \begin{pmatrix} U_R^* \\ U_C^* \end{pmatrix} - \frac{\beta^2}{2}\right) \quad (12)$$

where U_{R_j} and U_{C_i} are the coordinates of the samples in the standard normal space associated with the response and the capacity respectively, and U_R^* and U_C^* are the coordinates of the

MPP point in the standard normal space associated with response and the capacity respectively.

2.4 Reduced order modeling

Analyses of complex systems increasingly involve the use of finite element models aimed at solving a discretized version of the equilibrium equations of the problem. These finite element models usually need to be repeatedly queried to obtain the samples necessary for the calculation of the probability of failure by one of the previously described methods.

After space (and time) discretization, a finite element problem often involves a (set of) large linear system(s) of equations that need to be solved to obtain the finite element (FE) solution.

$$K(\mathbf{V}; \boldsymbol{\mu}) = \mathbf{F} \quad (13)$$

where $\mathbf{V} \in \mathbb{R}^n$ is the vector of the unknown state variables, $\boldsymbol{\mu} \in \mathbb{R}^p$ is a set of p parameters (material parameters, boundary conditions) so that $K : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$, n being the number of state variables and $\mathbf{F}$ being the vector of the loadings. In the context of reliability analysis we will consider that $\boldsymbol{\mu}$ is the vector of parameters that are considered to be uncertain. For example in heat transfer design problems $\mathbf{V}$ is the temperature field in the solid under consideration, $\boldsymbol{\mu}$ are material properties (conductivities, densities, etc) affecting the temperature field solution and $\mathbf{F}$ is the heat flux vector describing the boundary conditions. In structural design $\mathbf{V}$ is typically the displacement field in the solid under consideration, $\boldsymbol{\mu}$ are material properties (Young's modulus, Poisson's ratios, etc) affecting the displacement field solution and $\mathbf{F}$ is the vector of the forces describing the boundary conditions.

Let us assume that K is such that given any value of the set of parameters $\boldsymbol{\mu}$ a unique solution $\mathbf{V}=\mathbf{V}(\boldsymbol{\mu})$ exists. Very often K is also linear with respect to its first variable $\mathbf{V}$, such that the problem of Eq. (13) can be given by the following system of equations,

$$K(\boldsymbol{\mu})\mathbf{V} = \mathbf{F} \quad (14)$$

Note that from now on, $K(\boldsymbol{\mu})$ will be denoted simply as K but it always depends on the set of parameters $\boldsymbol{\mu}$.

Model order reduction [35] is a family of approaches that aims at significantly decreasing the computational burden associated with the solving of the system in Eq. (14), since industry problems typically involve millions of degrees of freedom thus requiring to solve a system with millions of equations. A particular class of model reduction techniques, called reduced basis approaches (or reduced order modeling by projection), aims at reducing the number of state variables of the model by projection on a certain basis. Accordingly, an approximation of the solution is sought in a subspace V of dimension m (with usually $m \ll n$), while enforcing the residual to be orthogonal to the same sub-space V . Typically, V is defined by a so called *reduced-basis* $\boldsymbol{\varphi} = \{\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_m\}$. Note that this approximate solution is different from surrogate modeling, since the reduced basis solution stems from solving the projected equilibrium equations and not from some sort of interpolation as is typical in surrogate modeling.

The problem of Eq. 14 is rewritten projected onto the reduced basis:

$$\boldsymbol{\varphi}^T K \boldsymbol{\varphi} \boldsymbol{\alpha} = \boldsymbol{\varphi}^T \mathbf{F} \quad (15)$$

Where $\boldsymbol{\varphi}$ is the $n \times m$ matrix formed by the vectors of the reduced basis and $\boldsymbol{\alpha}$ being the reduced state variables, that is the coefficients of vector $\mathbf{V}$ expressed in the reduced basis $\boldsymbol{\varphi}$.

It is important to realize why Eq. 15 is equivalent to a reduced order model of the initial problem of Eq. 14. Indeed, solving the problem of Eq. 14 typically involves the inversion of a large system of equations of size n , which is the number of degrees of freedom in the finite element problem. On the other hand, solving the reduced order model of Eq. 15 involves the inversion of a much smaller system of equations of size m , the size of the projected stiffness matrix $\boldsymbol{\varphi}^T K \boldsymbol{\varphi}$, which is equal to the dimensionality of the reduced basis m (typically $m \ll n$,

since for very reasonable accuracy m does usually not exceed a few dozens). Solving this reduced order model leads directly to α , the coefficients of the solution in the reduced basis.

The problem projected onto the reduced basis thus yields an approximate solution, $V_{RB} = \varphi\alpha$, whose accuracy can be quantified by measuring the following residual:

$$e_{rb}^2 = \frac{\|K\varphi\alpha - F\|^2}{\|F\|^2} \quad (16)$$

This residual is expressed in relative terms by the loading residual at the nominator over the actual loading at the denominator. The loading residual is the difference between the loading stemming from the approximate reduced basis solution and the actually applied loading on the boundaries. If the approximate solution $V_{RB} = \varphi\alpha$ was exact, the residual would be zero since the exact solution satisfies the equilibrium equations $KV = F$. Up to now, the subspace V on which the problem is projected, or more precisely one of its basis φ , was not specified and many different choices are possible for this projection. A method we propose for coupling reduced basis and reliability analysis will be detailed in the next section.

3 Reduced basis sampling strategy

3.1 Description

We propose in this section a generic approach for constructing and coupling the reduced basis construction with all five reliability analysis approaches investigated in this paper (FORM, standard Monte Carlo, separable Monte Carlo, importance sampling and combined separable Monte Carlo with importance sampling). Specifics of the implementation for each method will be provided in the next subsection.

All five reliability methods can be viewed as iterative approaches that need to calculate the solution of the equilibrium equations of Eq. 14 at each iteration. For FORM and importance

sampling, which involve an optimization loop for finding the most probable point (MPP) of failure, one iteration is considered to be an iteration of the optimization algorithm. For the approaches involving Monte Carlo sampling (all approaches but FORM), one iteration is considered to be the move from one sample to the next. An approach inspired from Gogu and Passieux [36] is proposed for constructing the reduced basis *on the fly* using calculations of the state variables at the previous iterations.

Let us consider iteration i of the reliability analysis process, where i state variable vectors $\mathbf{V}$ have already been calculated by solving the full equilibrium equations of Eq. 14. The subspace generated by these i previously calculated state variable vectors can form a reduced basis that will be used in the proposed approach for calculating the state variable at the next iteration (e.g. the next sampling point for sampling based approaches). That is, at iteration $i+1$ the approximate state variable vector will be calculated using the reduced order model of Eq. 15, which calculates the reduced state variables at the current iteration $i+1$ (and thus an approximation of the actual state variable vector) by solving the equilibrium equations projected on the subspace generated by the i previously calculated state variable vectors.

At iteration $i+2$ a new approximation of the state variable vector at this iteration can still be calculated using the reduced order model with the same subspace generated by the first i state variable vectors. This process can be applied until the approximate solution using the reduced order model is no longer sufficiently accurate, based on a threshold on the value of the residuals (Eq. 16).

Once the reduced order solution no longer satisfies this threshold a new full solution (inversion of the full system of Eq. 14) is computed again, solution which is also used for enriching the reduced basis. To enrich the existing reduced basis $\boldsymbol{\varphi}=\{\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_i\}$ the newly calculated solution $\mathbf{V}_{new}$ is orthogonalized as shown in Eq. 17 (Gram-Schmidt procedure), normalized (Eq. 18), and then added to the basis.

$$\tilde{\varphi}_{i+1} = V_{new} - \sum_{j=1}^i \langle V_{new}, \varphi_j \rangle \varphi_j \quad (17)$$

$$\varphi_{i+1} = \frac{\tilde{\varphi}_{i+1}}{\|\tilde{\varphi}_{i+1}\|} \quad (18)$$

where $\langle \bullet, \bullet \rangle$ denotes the L^2 scalar product.

To summarize the described process, a flowchart is provided in Figure 2.

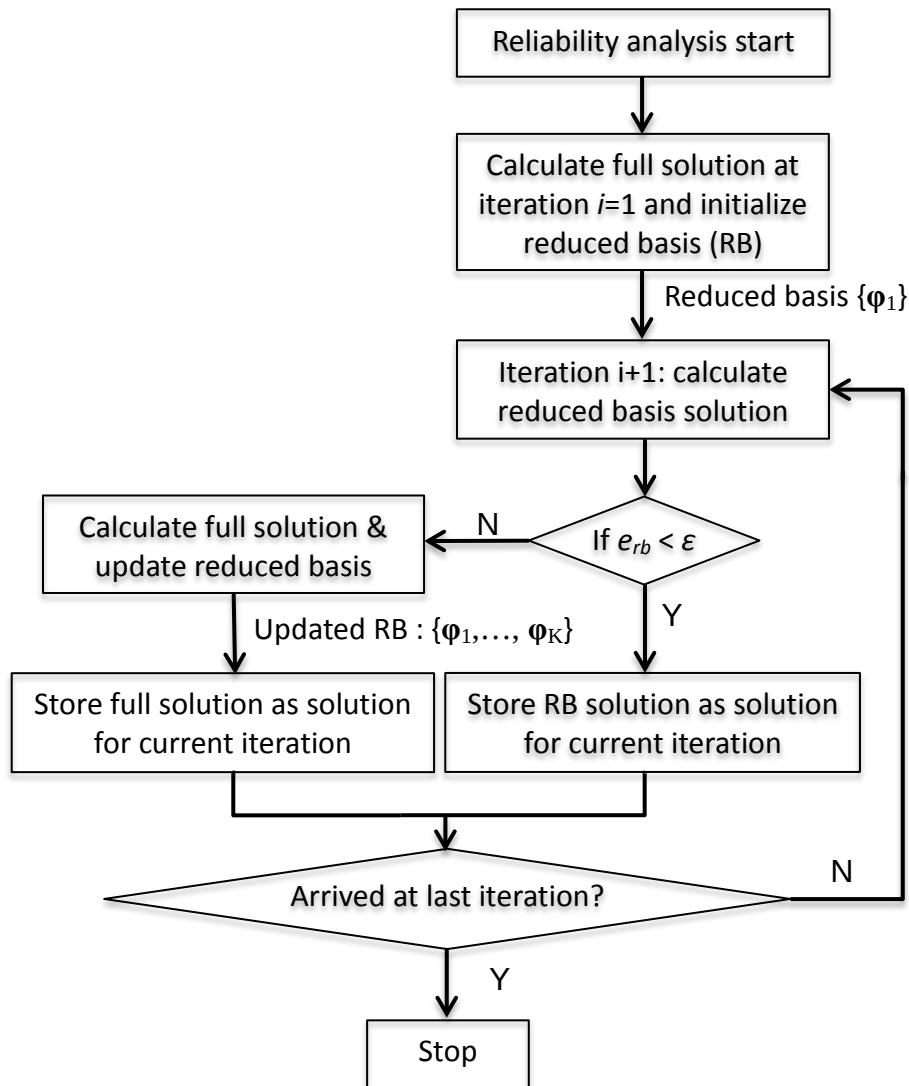


Figure 2: Flowchart of the reduced basis reliability analysis process using on the *on the fly reduced basis construction*

In the first step of this flowchart the reliability analysis starts and the full finite element solution (as opposed to reduced basis solution) needs to be calculated at the first iteration by solving the discretized finite element problem (Eq. 14). This full solution will also serve to initialize the reduced basis by normalizing the corresponding full solution vector. In subsequent iterations the reduced basis solution (cf. Eq. 15) is calculated first. If the residual e_{rb} corresponding to this reduced basis solution (cf. Eq. 16) is lower than the imposed threshold ε , then the reduced basis solution is used in place of the full solution at this iteration. Otherwise the full solution is calculated at this iteration and the reduced basis is updated by adding this full solution vector to the basis (cf. Eq. 17 and 18). If the current iteration is the last one in the reliability analysis (e.g. last sample) then the reliability analysis stops, otherwise it moves on to the next iteration and the same sequence repeats itself.

The advantage of the proposed approach is that it uses the reduced basis solution in place of the full solution whenever the threshold on the residuals is satisfied. This means that significant computational savings can be achieved since obtaining the reduced basis solution is much quicker than obtaining the full solution. Such a reduced basis approach appears particularly well suited for sampling based approaches since many samples lying in close vicinity to one another need to be evaluated. This close vicinity of the samples means that the reduced basis solution is likely to give reasonable results without a too frequent need for updating the reduced basis, thus having the potential to be computationally very efficient as compared to calculating the full solution for each sample.

3.2 Implementation specifics

We provide in this subsection two implementations of the proposed on the fly reduced basis construction method: one for the optimization framework that finds the most probable point (MPP) of failure and one for the advanced Monte Carlo sampling approaches.

Finding the MPP point consists in solving an optimization problem (cf. Eq. 2) involving a call to the solver of the equilibrium equation $KV=F$ of Eq. 14. The usual optimization algorithms used for solving this problem are iterative meaning that they will call the constraint function (and thus the solver of the equilibrium equations of Eq. 14) at each iteration. Algorithm 1 provided below replaces the call to the traditional solver at each iteration with an adaptive procedure, which at each iteration decides whether a reduced basis solution can be used depending on the residuals threshold imposed.

Algorithm 1: The code below should replace the line “ $V \leftarrow \text{solution of } KV = F$ ” in the reliability analysis process.

```

1:  if iteration = 1
2:       $V \leftarrow \text{solution of } K V = F$ 
3:      Add  $V / \text{norm}(V)$  to matrix  $\Phi$ 
4:  else
5:       $\alpha \leftarrow \text{solution of } (\Phi^T K \Phi) \alpha = \Phi^T F$ 
6:       $e_{rb} \leftarrow \text{norm}(K \Phi \alpha - F) / \text{norm}(F)$ 
7:      if  $e_{rb} > \varepsilon$ 
8:           $V \leftarrow \text{solution of } K V = F$ 
9:           $V_{ortho} \leftarrow V - \Phi (\Phi^T V)$ 
10:         Add  $V_{ortho} / \text{norm}(V_{ortho})$  to matrix  $\Phi$ 
11:      else
12:           $V \leftarrow \Phi \alpha$ 
13:      end if
14:  end if

```

In algorithm 1, lines 1-3 correspond to the reduced basis initialization with the full solution of the problem computed for the first iteration. Let us also recall that “ $A \leftarrow B$ ” means assign to variable A the value of expression B .

Once the reduced basis is initialized, the reduced basis solution is first calculated (line 5). The residual is calculated on line 6. If it is below a prescribed threshold ϵ , then the reduced basis solution is used in place of the exact solution (line 12) for the calculation of the constraint within the optimization algorithm seeking the MPP point. If the residual is higher than the threshold then the full solution is calculated (line 8) and the reduced basis updated by orthonormalizing the newly calculated solution.

A very similar procedure can be implemented for calculating the simulations for all the samples during advanced Monte Carlo sampling based procedures (traditional and separable Monte Carlo, importance sampling, etc...). The same algorithm 1 can be used here, the only difference being the definition of iteration. The sampling based procedures can be seen as iterative procedures as well in the sense that they calculate the simulations for each sample sequentially. For sampling based procedures iteration i would thus just mean the calculation of the simulation response for sampling point number i .

Within this context algorithm 1 can be seen as an adaptive procedure deciding for each sampling point whether only the reduced basis solution can be used or not depending on the prescribed residuals threshold. If the reduced basis solution is not accurate enough to be used, based on the residuals criterion of Eq. 16, then the full solution is calculated and the reduced basis enriched such as to improve future solutions using the reduced basis model.

Algorithm 1 can be applied both to the optimization algorithm for the search of the MPP as for the advanced Monte Carlo sampling algorithms. A specific point remains nevertheless to be clarified when importance sampling is coupled with the reduced basis approach. Importance sampling involves both a calculation of the MPP point and a subsequent sampling algorithm. When coupling importance sampling with the reduced basis approach two separate

reduced basis can of course be constructed, one for the calculation of the MPP and one for sampling. This is however somewhat inefficient since the reduced basis constructed initially for the MPP search can be reused also for the sampling. This is even more relevant since the sampling point will lie in majority relatively close to the MPP. We thus advise to reuse the same reduced basis for the sampling as for the MPP. Algorithm 1 continues to be used during the sampling, which means that if the available reduced basis does not lead to sufficiently accurate results, it will be enriched with new full solutions. In terms of practical implementation of the common reduced basis between MPP and sampling, we suggest the use of a global variable for the reduced basis.

It is now important to note that throughout the entire reduced basis reliability analysis procedures the only user prescribed parameter of the reduced basis procedure is the value of the residuals threshold. Below we provide some guidelines on how to choose this threshold for the reliability analysis algorithms presented.

First let us elaborate the meaning of this threshold. From Eq. 16 it is clear that the residuals on which the threshold applies is a relative error of the “loadings” vector predicted using the reduced basis solution compared to the actual “loadings” applied. In the context of reliability analysis we seek a quite accurate representation of the solution so a threshold higher than 10^{-3} (i.e. 0.1%) on the relative residual is usually not recommended. Of course, ideally, we would like to know the error on the state variable solution rather than on the “loadings”. In general however, assessing this error is almost as expensive as computing a full solution of the finite element problem, which defeats the purpose of the reduced order model here. It is possible nevertheless to draw some general guidelines based on few analyses. In sampling based reliability analyses the samples will lie in relatively close vicinity of the mean sampling point. Considering that the non-linearity of the solution in the uncertain parameters is not excessively high this means we can use the error computed at the mean sampling point to compute a residuals threshold that can be reused for the other sampling points as well [26].

Determining a reasonable value of this threshold can accordingly be done using the following procedure:

- Compute the full solution of the problem for the mean value point
- Construct a reduced basis using a low number of Monte Carlo samples (typically a few dozen) for various thresholds (e.g. 10^{-3} , 10^{-4})
- Compute the reduced order solution for the mean value point for the different thresholds
- Compare the full solution with the reduced basis solution at the mean value point and decide which threshold value is acceptable.

It is important to note that following this procedure does not mean that the reduced basis model will satisfy the prescribed accuracy only at the mean value point. The residual threshold will be verified at all the sampling points, including those in the tail of the distribution. The only assumption made through this procedure is that a residuals threshold at the mean value point is equivalent to the residuals threshold at a point in the tail of the distribution in terms of solution error. It is of course possible to find exceptions to this rule but in practice this assumption was found to be often verified.

4 Application

4.1 Application problem

The application considered here is a reliability analysis involving the heat transfer through the combustion chamber wall of a regeneratively cooled rocket engine [37],[38]. Note that the proposed approach cannot be applied to some of the typical analytical reliability benchmark problems. This is because our approach is specifically designed for reliability problems involving expensive finite element models. On the contrary, most reliability benchmark

problems involve relatively simple analytical test problems, which are thus not suited for the reduced basis modeling approach proposed here. In industry applications however, finite element models are frequent, and the coupling of the reduced basis approach with reliability methods is possible for a wide range of problems that can be formulated under the form of Eq. 14. This coupling is possible also when the problem is solved via a commercial finite element software. We selected here an example of such a finite element problem to illustrate the proposed approach and the possible computational savings.

A schematic of a typical regeneratively cooled liquid hydrogen (LH2) liquid oxygen (LOX) rocket engine is illustrated in Figure 3. The regenerative cooling takes place when the liquid hydrogen (LH2) at a temperature of 40K flows through cooling channels in the combustion chamber wall, before entering the injectors as shown in Figure 3. Failure, which manifests typically by the breakage of the cooling channel walls (cf. Figure 3, right image), is considered here when the maximum temperature of the cooling channel wall exceeds a design allowable T_{allow} .

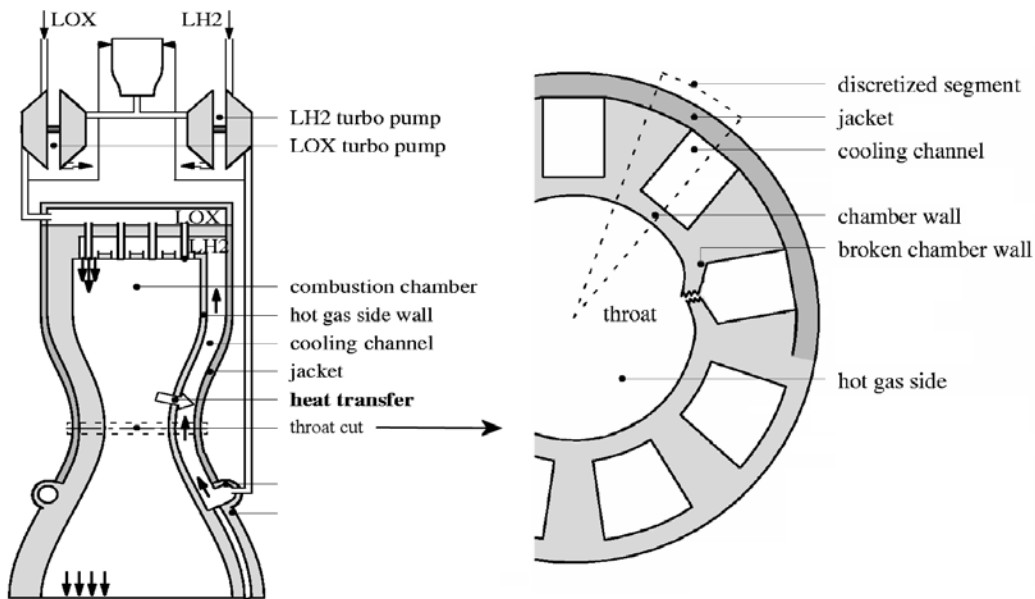


Figure 3: Schematic of a regeneratively cooled rocket engine combustion chamber. The drawings are not to scale.

Due to manufacturing constraints the combustion chamber wall is made of two different sections: an internal side usually made of a copper alloy in which the cooling channels are machined and an external jacket usually made of a Ni alloy (cf. Figure 3). Convective heat exchange can take place between the combustion chamber gases and the inner side of the combustion chamber wall, between the liquid hydrogen and the cooling channel side of the wall and between the outer environmental temperature and the external side of the combustion chamber wall. The corresponding heat transfer problem is thus parameterized by the following parameters: the conductivity of the inner side of the wall (k_{Cu}), the conductivity of the jacket (k_{Ni}), the temperature of the gases on the inner side of the combustion chamber (T_{hot}), the temperature on the outer side of the combustion chamber (T_{out}), the temperature of the cooling fluid (T_{cool}), the film convection coefficient on the inner side of the combustion chamber (h_{hot}), the film convection coefficient on the outer side of the combustion chamber (h_{out}) and the film convection coefficient on the cooling channel side (h_{cool}).

These eight parameters of the problem are considered not to be perfectly known, which is modeled by the parameters following the probability distributions given in Table 1. Furthermore the allowable for the maximum temperature T_{allow} follows a normal distribution with mean 750K and a coefficient of variation COV 7.5%.

To obtain the maximum temperature of the cooling channel wall a finite element model of the combustion chamber wall is constructed. By symmetry considerations only the discretized segment illustrated in Figure 3 (right side) is modeled. An in-house finite element solver coded in Matlab is used here. The boundary conditions and finite element mesh are illustrated in Figure 4.



Figure 4: Schematic of a regeneratively cooled rocket engine combustion chamber. The drawings are not to scale.

Table 1. Distributions of the parameters of the thermal problem.

Parameter	k_{Cu}	k_{Ni}	T_{hot}	T_{out}	T_{cool}	h_{hot}	h_{out}	h_{cool}
	(W/(mK))	(W/(mK))	(K)	(K)	(K)	(kW/(m ² K))	(kW/(m ² K))	(kW/(m ² K))
Distribution	normal	normal	uniform	uniform	uniform	uniform	uniform	uniform
Mean	310	75	900	293	40	31	6	250
COV or	2%	2%						
Half-range			10%	5%	5%	10%	5%	10%

Our aim is to determine the probability of failure P_f , i.e. the probability that the response, consisting in the maximum temperature of the cooling channel calculated by finite elements, exceeds the capacity, consisting in the temperature allowable T_{allow} . To calculate this probability of failure P_f we will use in section 4.2 the five approaches described previously without reduced basis modeling, while in section 4.3 we will provide the results for these same methods when using the reduced basis method. Our aim is to compare the computational costs of the different methods needed to achieve a given coefficient of variation of the probability of failure.

4.2 Results of the reliability analysis methods without reduced basis modeling

We first compute the results of the standard Monte Carlo method with 10^6 simulations and the FORM approach. The probability of failure found with standard MC was $7.60 \cdot 10^{-3}$, which is considered to be the true probability of failure, while the probability of failure computed with FORM was $1.64 \cdot 10^{-2}$. The FORM solution turns out to be quite poor here and this is due to the linearity approximation in FORM which is not really satisfied for this problem. The poor performance of FORM on such non-linear problems is what justifies the use of various sampling based approaches, such as the ones described in section 2.

Next we sought to compare the computational efficiency of the considered sampling based approaches (standard Monte Carlo method, importance sampling, separable Monte Carlo and their synergetic combination) on this problem, while achieving a coefficient of variation (COV) of roughly 4% on the probability of failure. For the standard Monte Carlo approach, the COV was computed analytically using Eq. 6. For the remaining sampling based approaches the COV was computed by repeating the entire procedure 1000 times. At each repetition new samples for the response and capacity were drawn.

We provide in Table 2 the results of the standard Monte Carlo (MC), importance sampling (IS), separable Monte Carlo (SMC) and the combined approach of separable Monte Carlo with importance sampling (ImpSMC). The second column provides the number of samples used for each approach. The third column provides the COV of the probability of failure. On average over the 1000 repetitions the true probability of failure of $7.60 \cdot 10^{-3}$ was accurately found. Finally in the fourth column we provide the computational cost of the approach, normalized such that the cost of the standard Monte Carlo is equal to 1.

Table 2. Results of different methods for calculating the probability of failure for the combustion chamber wall problem

Method	Number of samples	Coefficient of variation	Normalized computational time
MC	$N = 80000$	4.04%	1
IS	$N = 5000$	3.54%	0.063
SMC	$N = 13200$ $M = 15000$	4.05%	0.16
ImpSMC	$N = 1200$ $M = 5000$	3.79%	0.014

note that : MC = Monte Carlo simulation

IS = Importance sampling

SMC = Separable Monte Carlo simulation

ImpSMC = Separable Monte Carlo with importance sampling

4.3 Results of the reliability analysis methods with reduced basis modeling

The considered sampling based approaches were also implemented using the reduced basis (RB) coupling described in section 3. The same thermal reliability problem described in section 4.1 was solved and the results in terms of COV and computational cost are presented in Table 3. Note that the computational cost is normalized here with respect to the computational cost of the Monte Carlo approach without reduced basis modeling (which was computed in section 4.2 and has in Table 2 a computational cost of 1). On average over the 1000 repetitions the true probability of failure of $7.60 \cdot 10^{-3}$ was again accurately found.

Table 3. Results of the different proposed methods for calculating the probability of failure

Method	Number of samples	Coefficient of variation	Normalized computational time
RB-MC	$N = 80000$	4.08%	0.17
RB-IS	$N = 5000$	3.54%	0.011
RB-SMC	$N = 13200$	4.08%	0.028

	$M = 15000$		
RB-ImpSMC	$N = 1200$ $M = 5000$	3.87%	0.0026

note that : RB-MC = Monte Carlo simulation with the proposed reduced basis coupling
RB-IS = Importance sampling with the proposed reduced basis coupling
RB-SMC = Separable Monte Carlo simulation with the proposed reduced basis coupling
RB-ImpSMC = Separable Monte Carlo with importance sampling with the proposed reduced basis coupling

By comparing Tables 2 and 3, we note that, for roughly a same coefficient of variation ($\sim 4\%$) on the probability of failure, significant computational cost reductions are achievable by the proposed approaches compared to conventional sampling methods without reduced basis. Note that the normalization factor for calculating the normalized time is the same in Table 2 and in Table 3, thus allowing direct comparison between the two tables. For example we can see that the proposed reduced basis Monte Carlo approach (RB-MC in Table 3) is 5.88 times more efficient than the traditional Monte Carlo approach (MC in table 2).

It is also worth noting that for a same approach, the differences on P_f between using or not the reduced basis are very small: the same true probability of failure is found on average over the 1000 repetitions and the COV is the same up to two significant digits. This means that the reduced basis solution is very accurate, which is notably due to the low residual threshold that was selected $e_{rb} = 10^{-4}$. In spite of the very good accuracy, the reduced basis approach allowed to achieve significant computational cost savings. The most drastic reduction is achieved while compounding the different methods: while obtaining roughly the same COV of $\sim 4\%$, the reduced basis combined ImpSMC approach is more efficient by a factor of 384 compared to the conventional Monte Carlo approach (without reduced basis modeling).

The first three temperature modes for the RB-MC method, i.e. the first three vectors of the reduced basis, are represented in Figure 5. Note that the basis vectors for the RB-MC method are presented but the first three basis vectors are very similar, independently of the chosen

method (RB-MC, RB-IS, RB-SMC or RB-ImpSMC). These three modes are the ones that contribute most to the reduced basis representation and are thus responsible for the significant computational efficiency improvements obtained.

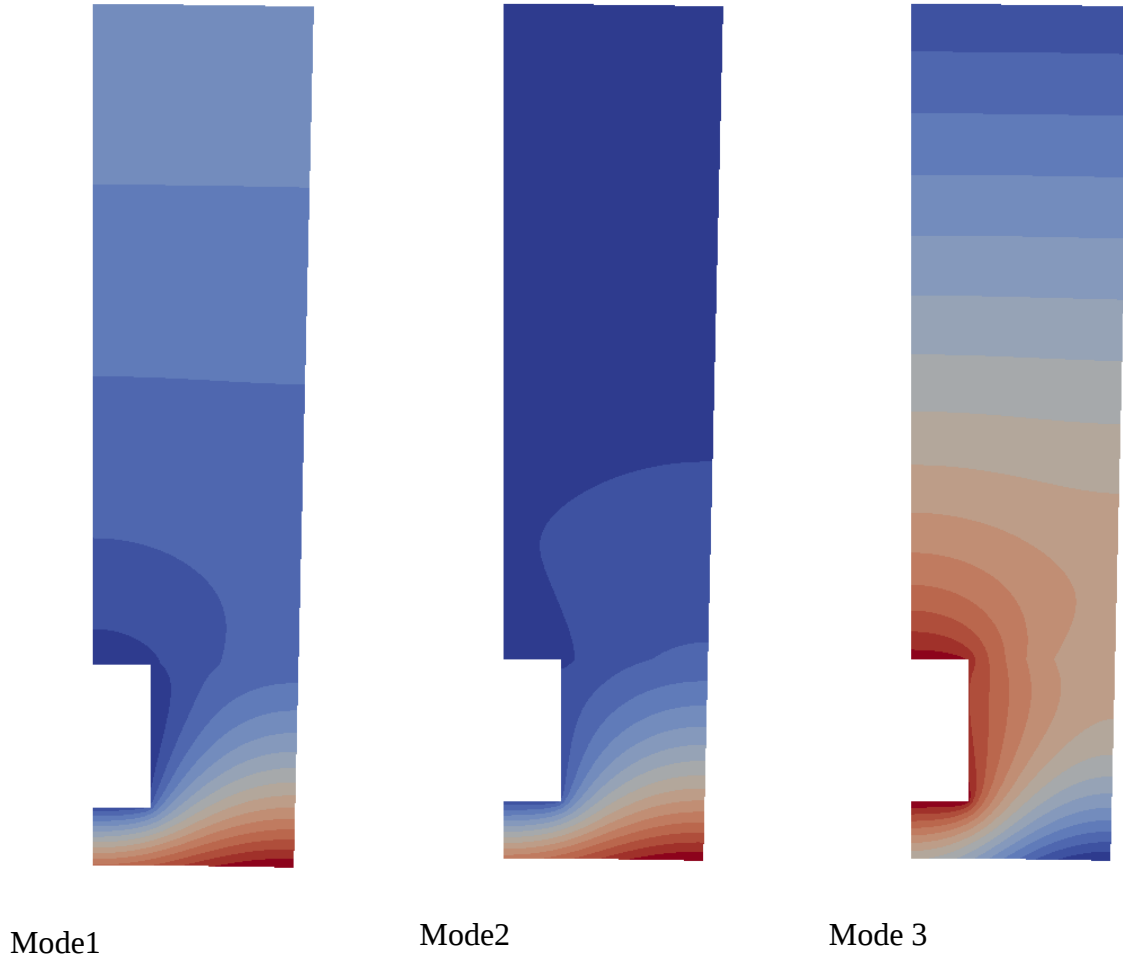


Figure 5. First three temperature modes of the reduced basis approaches

The number of total basis vectors used in each method is provided in Table 4. This number is thus the number of full simulations that were carried out during the entire reliability analysis. The number of reduced basis solutions that were calculated during the entire reliability analysis is also provided in Table 4. For example for the reduced basis importance sampling (RB-IS) approach only 12 full solutions of the finite element problem while 4988 reduced

basis solutions were calculated. The large number of reduced basis solutions that are possible explains the significant computational savings achieved with the reduced basis approach.

Table 4. Number of full and reduced basis solutions for the different reduced basis methods

Method	Number of reduced basis solutions	Number of full FE solutions (= number of basis vectors)
RB-MC	79987	13
RB-IS	4988	12
RB-SMC	13188	12
RB-ImpSMC	1189	11

note that : RB-MC = Monte Carlo simulation with the proposed reduced basis coupling

RB-IS = Importance sampling with the proposed reduced basis coupling

RB-SMC = Separable Monte Carlo simulation with the proposed reduced basis coupling

RB-ImpSMC = Separable Monte Carlo with importance sampling with the proposed reduced basis coupling

4.4 Discussion

The speed-ups achieved by using the reduced basis approach in the various sampling methods vary between a factor of 5 and 6. The speed-up that can be obtained on different problems is affected by various factors, notably the assembly time of the matrix K and the ratio of the computational cost of calculating a reduced basis solution over the cost of a calculating a full solution.

The assembly could be implemented quite efficiently on this problem with the assembly time representing less than $1/10^{\text{th}}$ of the inversion time. On other problems where the assembly time can represent a larger fraction of the total computational time there is always the possibility to parallelize the assembly on multiple core machines. This parallelization is indeed straight forward through a parallel *for* loop. Note also that the relative part of the assembly time decreases with the problem size (i.e. the number of degrees of freedom DOF).

The ratio of the computational cost of a reduced basis solution over the cost of a full solution, which is the main factor affecting the efficiency of the proposed approaches, is problem dependent. It appears that the most significant factor affecting this ratio is the size of the problem as well. The larger the size of the problem the smaller this ratio gets. Together with parallel assembly of the K matrix this implies that for large size problems the approach has the potential of great computational cost savings with the predominant computational cost being only that of computing the full solutions required for constructing the reduced basis. For problems with a large number of degrees of freedom (e.g. over a million) the proposed approaches have the potential to lead cost savings by more than an order of magnitude [39].

Note that as described in section 3.2 the threshold value on the residuals of the reduced basis model had to be selected. This threshold defines the trade-off between reduced basis solution accuracy and reduced basis size (and thus reduced basis solution efficiency). To determine the appropriate threshold we followed the iterative approach described in section 3.2. We tested three different values of the threshold ($e_{rb} = 10^{-2}, 10^{-3}, 10^{-4}$) with a small size of 20 samples in order to determine the accuracy of the reduced order model. On this problem we found that the reduced basis size did not vary much with the value of the threshold thus not significantly affecting the reduced basis solution cost with either of these three threshold values. Accordingly we chose the lowest threshold value ($e_{rb} = 10^{-4}$) leading to the most accurate reduced basis solution.

The number of full solutions that had to be calculated during the reliability analyses was found to be quite low here, with only a dozen full solutions being required, while all the remaining (several thousands) sampling points could be calculated with a more cost efficient reduced basis solution. This can be explained by the fact that in typical reliability estimations most sampling points are in relatively close vicinity, thus allowing reduced basis solutions with relatively low basis size to be very efficient. Note that the uncertainties chosen were not particularly low for the application considered (up to 10% COV), so the reduced basis

solution is found to work well over the entire domain where the sampling points were tightly clustered together. Obtaining efficient reduced basis models that involve a low-dimensional basis (up to a few dozen basis vectors) are quite frequent and have been effectively used in a wide range of applications (modelling different physics) as has been pointed out by the recent survey by Benner et al. [24]. The finite element problem does however need to be one which can be written under the form provided in Eq. 14. While there are many problems in a variety of domains (heat transfer, structural mechanics, electromagnetics) that do verify this form there are also some type of problems (e.g. non-linear, contact type problems) which do not verify it and where the proposed reduced basis approach cannot be directly applied. Note that the proposed approach is generally applicable even with commercial finite element software modeling problems of the form of Eq. 14, since most of the commercial codes allow the extraction of the matrix K and loading vector F (cf. Eq. 14). The user thus does not have to write his own finite element code but he can use existing software and simply implement Algorithm 1 for the reduced basis construction within the code he uses for reliability analysis.

5 Conclusions

This article explored the coupling between reduced basis modeling and various sampling based reliability formulations. Coupling strategies and their corresponding algorithms have been proposed for standard Monte Carlo, importance sampling, separable Monte Carlo and the combined importance sampling with separable Monte Carlo approach. Note that while these four sampling approaches have been chosen for illustrating this coupling, the approach has the potential to be applied to the majority of the other sampling based reliability estimation methods. The proposed approach thus should not be seen as a method competing with existing reliability estimation methods but rather as a possible complement, allowing

further computational savings when the problem under consideration involves certain types of expensive finite element models.

The four sampling based approaches selected here have been compared on a reliability analysis problem involving a heat transfer model of a rocket engine combustion chamber wall. We found that the reduced basis approach coupled with the combined importance sampling separable Monte Carlo approach was computationally more efficient by a factor of 384 compared to the standard Monte Carlo approach without reduced basis modeling for this problem. Reduced basis modeling alone accounted for a factor of about 6 out of the total speed up achieved by this method. Given the workings of the reduced basis approach this speed-up further increases with the size of the finite element problem making these approaches particularly appealing for reliability analyses involving very large size problems (millions of degrees of freedom). Future work and further possible improvements include investigating the coupling of reduced basis methods with other reliability analysis methods such as subset simulation and adaptive sampling surrogate based approaches (e.g., EGRA [10], adaptive target region [11], AK-MCS [12], adaptive SS [13], SUR [14]) as well as investigating adaptive residual threshold selection within the reduced basis modeling.

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