Improving the efficiency of large scale topology optimization through on the fly reduced order model construction

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Abstract. Topology optimization of large scale structures is computationally expensive, notably due to the cost of solving the equilibrium equations at each iteration. Reduced order models by projection, also known as reduced basis models, have been proposed in the past for alleviating this cost. We propose here a new method for coupling reduced basis models with topology optimization to improve the efficiency of topology optimization of large scale structures. The novel approach is based on constructing the reduced basis on the fly, using previously calculated solutions of the equilibrium equations. The reduced basis is thus adaptively constructed and enriched, based on the convergence behavior of the topology optimization. A direct approach and an approach with adjusted sensitivities are described and their algorithms provided. The approaches are tested and compared on various 2D and 3D minimum compliance topology optimization benchmark problems. Computational cost savings by up to a factor of 12 are demonstrated using the proposed methods.

Keywords: topology optimization, reduced order models, reduced basis, on the fly construction

1 Introduction

Over the last two decades topology optimization has undergone a rapid period of growth both in industry and academia spurred by a large number of theoretical, practical and algorithmic developments [1]-[3]. One of the challenges in topology optimization is dealing with large scale problems that can involve millions of degrees of freedom. Indeed, in the classical nested approach, at each iteration of the optimization process the equilibrium equations, characterizing the structure, need to be solved. Solving these equations can then quickly become the dominant computational expense in the topology optimization process, as has been pointed before [3],[14]. For the density based approach using a classical optimality criteria design update this cost can represent about 90% of the total computational cost already for a medium size problem with 100 000 design variables, and this fraction further increases with the size of the problem.

One way that has been recently explored for decreasing this cost is the use of reduced order models. Reduced order models provide an approximation of the solution of the exact equilibrium equations with drastically reduced computational cost, and can thus replace the exact solution at appropriately selected iterations of the topology optimization. They have gained much interest in recent years in various domains, be it in analysis [4]-[7] or optimization [8]-[11]. A particular type of reduced order model is the so called reduced order model by projection or reduced basis model, in which the equilibrium equations are being solved projected on a certain basis, which is usually of much lower dimension than the size of the system of equilibrium equations themselves. A central question in reduced basis methods is how to construct the reduced basis. Several such approaches have been proposed in the literature aimed at enhancing the efficiency of topology optimization. Wang et al. [12] and Amir et al. [13] used Krylov subspaces to construct the reduced basis. Amir et al. [14] also proposed the construction of a reduced basis using the combined approximations method.

Yoon [15] used eigenmodes and Ritz vectors to construct the reduced basis in topology optimization for vibration response.

In this paper a new approach for constructing the reduced basis in the context of topology optimization is proposed, inspired from Gogu and Passieux [16]. The approach uses previously calculated solutions of the equilibrium equations to construct the reduced basis *on the fly*, at limited additional computational cost. Two algorithms implementing this approach for minimum compliance structural topology optimization problems are proposed and their accuracy and efficiency are investigated on various 2D and 3D problems.

The rest of the paper is organized as follows. In section 2 the theoretical formulation is reviewed. First the topology optimization background is presented, followed by the reduced order modeling and the reduced basis construction. In section 3, two topology optimization approaches implementing the *on the fly* reduced basis construction are presented. The first one is a simple, straight forward implementation of the reduced basis construction. In section 4, numerical investigations are carried out to assess the accuracy and efficiency of the proposed approaches.

2 Theoretical formulations

2.1 Topology optimization background

In this study, a classical density based approach to topology optimization will be used [17], [18]. This approach uses the density of each element of the structure, density which in turns determines the element's Young's modulus. Using a modified solid isotropic material with penalization (SIMP) model [18] the density of an element can be expressed as:

$$E_e(\rho_e) = E_{\min} + \rho_e^p (E_{no\min al} - E_{\min})$$
⁽¹⁾

Where E_e is the Young's modulus of an element having a density ρ_e . The nominal Young's modulus of the material to be used is $E_{nominal}$ while E_{min} is a very small positive Young's modulus value, imposed in order to avoid numerical issues related to elements with zero stiffness.

The topology optimization problem is then expressed as a compliance minimization problem, i.e. the optimization formulation seeks to find the density distribution over all the elements that minimizes the work done by the external forces under prescribed loadings, boundary conditions and material volume fraction to be used. The mathematical formulation of this optimization problem can be expressed as:

$$\min_{\rho} (c(\rho)) = F^{T}U = U^{T}KU$$

$$s.t.: \sum_{e=1}^{N} v_{e}\rho_{e} \leq V$$

$$0 \leq \rho_{e} \leq 1 \qquad e = 1,...,N$$

$$KU = F$$
(2)

Where c is the compliance of the structure, ρ the vector of design variables consisting of the individual element densities ρ_e , F the external forces vector, U the displacements vector, K the global stiffness matrix of the structure, v_e the volume of an element and V the maximum prescribed volume for the entire structure. The dimension of the displacement and forces vector is denoted by n.

In order to avoid mesh-dependency and checkerboard patterns when solving the topology optimization problem various filtering approaches are typically applied [19]. In this study a density filter is used [20],[21].

For topology optimization of large scale structures the bulk of the computational cost comes from the requirement to compute at each step of the optimization the solution of the equilibrium equations:

$$KU = F \tag{3}$$

Computing this solution for large scale problems involves the inversion of a very large system of equations that can consist of up to millions of degrees of freedom.

2.2 Reduced order modeling

Model order reduction [23] is a family of approaches that aims at significantly decreasing the computational burden associated with the inversion of the system in Eq. (3). A particular class of model reduction techniques, called reduced basis approaches (or reduced order modeling by projection), aims at reducing the number of state variables of the model by projection on a certain basis. Accordingly, an approximation of the solution is sought in a subspace V of dimension *m* (with usually *m*<<*n*), while enforcing the residual to be orthogonal to the same sub-space V. Typically, V is defined by a so called *reduced-basis* $\Phi = {\Phi_1, ..., \Phi_m}$. The initial problem of Eq. 3 is rewritten projected onto the reduced basis:

$$\Phi^T K \Phi \boldsymbol{\alpha} = \Phi^T \boldsymbol{F} \tag{4}$$

where α are the reduced state variables, that is the coefficients of vector U expressed in the reduced basis Φ .

It is important to realize why Eq. 4 is equivalent to a reduced order model of the initial problem of Eq. 3. Indeed, solving the problem of Eq. 3 typically involves the inversion of a large system of equations of size *n*, the size of the stiffness matrix *K*, which for large scale problems can easily reach millions. On the other hand solving the reduced order model of Eq. 4 involves the inversion of a much smaller system of equations of size *m*, the size of the projected stiffness matrix $\Phi^T K \Phi$, which is equal to the dimensionality of the reduced basis *m* (typically *m*<<*n*, since for very reasonable accuracy *m* does usually not exceed a few dozens). Solving this reduced order model leads directly to *a*, the coefficients of the solution in the reduced basis.

The problem projected onto the reduced basis thus yields an approximate solution $U_{RB} = \Phi \alpha$, whose accuracy can be quantified by measuring the following residual:

$$e_{rb}^{2} = \frac{\left\| K \Phi \boldsymbol{\alpha} - \boldsymbol{F} \right\|^{2}}{\left\| \boldsymbol{F} \right\|^{2}}$$
(5)

This residual is expressed in relative terms by the forces residual at the nominator over the actual forces at the denominator. The forces residual is the difference between the forces stemming from the approximate reduced basis solution and the actually applied forces. If the approximate solution $U_{RB} = \Phi \alpha$ was exact, the residual would be zero since the exact solution satisfies the equilibrium equations KU = F.

Note that since this residual is expressed in terms of a norm, several combinations of α could in theory lead to the same residual. In practice this seems seldom to be an issue, since the α coefficients have a strong physical foundation, stemming from the solution of the projected physical equations. This tends to avoid numerical instabilities and ill conditioning which could have caused issues in using the residual error criterion. Note also that the criterion of Eq. 5 is expressed in terms of the residual of the forces vector. Another option would have been to seek the error on the displacement vectors. The advantage of such an alternative error criterion is that it seeks to estimate the error directly on the quantity of interest. The drawback however is that such an error criterion is much more expensive to estimate than the residuals on the forces. Approximate solutions for estimating this error exist [24], but the residuals criterion of Eq. 5 still remains significantly cheaper to estimate and appears to provide reasonable results for a wide variety of problems.

2.2 Construction of the reduced basis

Up to now, the subspace V on which the problem is projected, or more precisely one of its basis Φ , was not specified and many different choices are possible for this projection.

In the context of topology optimization Wang et al. [12] and Amir et al. [13] proposed methods based on the use of Krylov subspaces. Amir et al. [14] proposed in a different approach the construction of a reduced basis using the combined approximations method. Yoon [15] used eigenmodes and Ritz vectors for the reduced basis in topology optimization for vibration response.

In this paper a new approach is proposed for constructing the reduced basis *on the fly* using previous calculations of the state variables (i.e. displacement vector).

Let's consider iteration i of a classical topology optimization process where i displacement vectors have already been calculated (one at each iteration) by inverting the full equilibrium equations of Eq. 3. The subspace generated by these i previously calculated displacement vectors can form a reduced basis that will be used in the proposed approach for calculating the displacement vector at the next iteration. That is, at iteration i+1 the approximate displacement vector will be calculated using the reduced order model of Eq. 4, which calculates the reduced state variables at the current iteration i+1 (and thus an approximation of the displacement vector) by solving the equilibrium equations projected on the subspace generated by the i previously calculated displacement vectors. At iteration i+2 a new approximation of the displacement vector at this iteration can still be calculated using the reduced order model with the same subspace generated by the first i displacement vectors. This process can be applied until the approximate solution using the reduced order model is no longer sufficiently accurate, based for example on a threshold on the value of the residuals (Eq. 5).

Once the reduced order solution no longer satisfies this threshold a new full solution (inversion of the full system of Eq. 3) is computed again, solution which is also used for enriching the reduced basis. To enrich the existing reduced basis $\Phi = \{\Phi_1, ..., \Phi_i\}$ the newly calculated solution U_{new} is orthogonalized as shown in Eq. 6 (Gram-Schimdt procedure), normalized, then added to the basis.

$$\widetilde{\Phi}_{i+1} = U_{new} - \sum_{j=1}^{i} \left\langle U_{new}, \Phi_j \right\rangle \Phi_j$$
(6)

Where $\langle \bullet, \bullet \rangle$ denotes the L^2 scalar product.

Note that while the reduced basis uses solutions from previous iterations, these solutions are only added to the basis if the approximation is too poor (as measured by the residual error criterion). Furthermore any new solution is only added to the basis after orthonormalization by the Gram Schimdt procedure, meaning that only the component of the solution which is orthogonal to the already existing basis vectors is actually added to the basis.

Note also that the construction of the reduced basis using previously calculated solutions ensures relatively quick convergence. Indeed, unlike for small dimensional problems, a reduced basis formed with randomly generated designs will not converge here due to the very high dimensionality of the problem (up to millions of input design variables).

Two approaches based on this *on the fly reduced basis construction* are proposed in the next section.

3 On the fly reduced basis topology optimization

3.1 Direct iterative approach

The proposed direct iterative approach works with a fixed, user defined, reduced basis size, denoted N_b . The reduced basis is initialized using the first N_b iterations of the topology optimization process. Indeed at each iteration the full equilibrium equations of Eq. 3 are solved and the corresponding solution displacement vector obtained. These N_b solution vectors form the initial subspace of the reduced basis. In order to obtain the basis vectors of this subspace, the N_b solution vectors need to be orthonormalized. To achieve this, the first solution vector U_I is first normalized:

$$\Phi_1 = \frac{U_1}{\|U_1\|} \tag{7}$$

The subsequent basis vectors are obtained by the Gram Schmidt orthogonalization and subsequent normalization as shown in Eqs. 8 and 9:

$$\widetilde{\Phi}_{i+1} = U_{i+1} - \sum_{j=1}^{i} \left\langle U_{i+1}, \Phi_j \right\rangle \Phi_j \quad \text{for } i=1, \dots, N_b-1$$
(8)

$$\Phi_{i+1} = \frac{\widetilde{\Phi}_{i+1}}{\left\|\widetilde{\Phi}_{i+1}\right\|} \qquad \text{for } i=1,\dots,N_b-1 \tag{9}$$

The reduced basis being now initialized, the reduced basis solutions obtained via Eq. 4 will from now on be used at each of the subsequent topology optimization iterations, replacing the call to the full solution of the equilibrium equation (Eq. 3). In order to maintain appropriate convergence of the topology optimization, the quality of the reduced basis approximation at each iteration needs to be checked using the value of the residuals (Eq. 5). If the value of the residual e_{rb} is lower than a user defined threshold ε , then the reduced basis solution is considered to be accurate enough to be used. If this criterion is verified, the approximate reduced basis solution will be used in the topology optimization process (e.g. for calculating sensitivities, objective function, etc) as if it was the exact solution, i.e. the approximate solution replaces the exact solution for the current optimization iteration. In the case where e_{rb} $> \varepsilon$ the reduced basis solution is not considered accurate enough and a new full solution obtained by inverting the full equilibrium equations (Eq. 3) is calculated. The reduced basis is then updated using this new full solution vector. Since we work at fixed reduced basis size the oldest vector in the reduced basis is discarded. Finally, in order to add the new solution vector to the reduced basis, it is orthogonalized (Eq. 6) and normalized. The same process can then start over.

To summarize this process, a flowchart is provided in Figure 1.



Figure 1: Flowchart of the reduced basis topology optimization process using on the *on the fly* reduced basis construction

The advantage of the proposed approach over the usual topology optimization process is that after the reduced basis initialization, at each iteration of the optimization process the reduced basis solution of the equilibrium equations is first calculated. If the residuals criterion is verified then the approximate reduced basis solution is used at this iteration in place of the exact solution, meaning that significant computational savings can be achieved since obtaining the reduced basis solution is much quicker than obtaining the full solution.

From an implementation point of view the proposed approach turns out to be quite simple, since one can take the topology optimization code of his preference and just replace the line of the code that calculates the solution of the equilibrium equation (typically this line calculates the solution of KU=F) with the Algorithm 1 presented below.

```
Algorithm 1: Direct iterative approach algorithm. The code below should replace the line "U \leftarrow solution of KU = F" in a usual topology optimization algorithm.
```

```
1:
       if iteration = 1
2:
              U \leftarrow \text{solution of } K U = F
3:
              Add U/\operatorname{norm}(U) to matrix \Phi
       else if 1 < iteration \le N_b
4:
5:
              U \leftarrow \text{solution of } K U = F
              U_{ortho} \leftarrow U - \Phi (\Phi^T U)
6:
7:
              Add U_{ortho} / norm(U_{ortho}) to matrix \Phi
8
       else
             \alpha \leftarrow \text{solution of } (\Phi^T K \Phi) \alpha = \Phi^T F
9:
```

10: $e_{rb} \leftarrow \operatorname{norm}(K \Phi \alpha - F) / \operatorname{norm}(F)$ 11: if $e_{rh} > \varepsilon$ 12: Remove oldest basis vector from matrix Φ $U \leftarrow \text{solution of } K U = F$ 13: $U_{ortho} \leftarrow U - \Phi (\Phi^T U)$ 14: 15: Add U_{ortho} / norm(U_{ortho}) to matrix Φ 16: else $U \leftarrow \Phi \alpha$ 17: 18: end if 19: end if

In algorithm 1, lines 1-7 correspond to the reduced basis initialization. From iteration 1 to N_b of the topology optimization, the full solutions of the equilibrium equation KU=F are calculated. Note that here, *iteration* denotes the number of the iteration (or cycle) of the topology optimization routine and that the matrix of the basis vectors Φ is the empty matrix before the first iteration. It is also recalled that " $A \leftarrow B$ " means assign to variable A the value of expression B.

Once the reduced basis initialized, the reduced basis solution is first calculated (line 9). The residual is calculated on line 10. If it is below the threshold ε , then the reduced basis solution is used in place of the exact solution (line 17) for the remaining calculations (sensitivity, objective function) of the current topology optimization iteration. If the residual is higher than the threshold then the full solution is calculated (line 13) and the reduced basis updated.

In usual topology optimization codes, the line that is being replaced here by Algorithm 1 provides the exact displacement vector solution U satisfying the equilibrium equations. Algorithm 1 also provides a displacement vector U as an output. This displacement vector is sometimes the exact solution and sometimes only the reduced basis solution, if the reduced basis approximation is found to be acceptable in terms of accuracy. Since the reduced basis solution is much cheaper to evaluate than the full solution, the frequency with which only the reduced basis solutions can be used will determine the computational savings of the proposed method.

Note that Algorithm 1 has only two parameters that need to be defined by the user beforehand: the size of the reduced basis N_b and the threshold ε on the residuals. The effect of the choice of these parameters will be investigated in subsequent sections of this paper.

3.2 Iterative approach with sensitivity adjustment

The approach that was just presented has the advantage of being simple and straight forward to implement. Nevertheless it makes the assumption that the exact solution can be replaced by the reduced basis solution at some iterations of the topology optimization process without negatively affecting the convergence or the solution of the topology optimization. As will be shown by the numerical investigations in section 4, this assumption can be reasonable depending on the choice of the user defined parameters. Nevertheless a different algorithm, inspired from Amir et al. [14] is also provided in the present subsection, taking into account the fact that only an approximate solution is being sometimes used.

In order to update the topology at each iteration, the topology optimization process uses the sensitivity of the objective function (compliance) with respect to density variations. The classical expression of the sensitivity is:

$$\frac{\partial c}{\partial \rho_e} = -U^T \frac{\partial K}{\partial \rho_e} U \tag{10}$$

When using algorithm 1 presented in the previous subsection, the displacement vector U is sometimes an approximation (the reduced basis solution $U_{RB} = \Phi \alpha$), this expression of the sensitivity is thus only approximate. In the present subsection we seek to correct the value of the sensitivity by using the adjoint method [22] to take into account the approximation error. For this purpose we define the following modified objective function, by adding additional terms that are equal to zero:

$$c(\rho_e) = \alpha^T \Phi^T K \Phi \alpha - 2\tilde{\alpha}^T \left(\Phi^T K \Phi \alpha - \Phi^T F \right) - \sum_{i=1}^{N_b} \lambda_i^T \left(K_i U_i - F \right)$$
(11)

Where $\tilde{\alpha}$, λ_i (with *i*=1,...,N_b) are adjoint variables and K_i and U_i are the stiffness matrix and displacement fields corresponding to the ith basis vector.

The modified expression of the sensitivity is obtained by differentiating Eq. 11 with respect to the elemental density:

$$\frac{\partial c}{\partial \rho_{e}} = 2 \frac{\partial \alpha^{T}}{\partial \rho_{e}} \Phi^{T} K \Phi \alpha + 2 \alpha^{T} \frac{\partial \Phi^{T}}{\partial \rho_{e}} K \Phi \alpha + \alpha^{T} \Phi^{T} \frac{\partial K}{\partial \rho_{e}} \Phi \alpha$$

$$+ 2 \widetilde{\alpha}^{T} \frac{\partial \Phi^{T}}{\partial \rho_{e}} F - 2 \widetilde{\alpha}^{T} \frac{\partial \Phi^{T}}{\partial \rho_{e}} K \Phi \alpha - 2 \widetilde{\alpha}^{T} \Phi^{T} \frac{\partial K}{\partial \rho_{e}} \Phi \alpha$$

$$- 2 \widetilde{\alpha}^{T} \Phi^{T} K \frac{\partial \Phi}{\partial \rho_{e}} \alpha - 2 \widetilde{\alpha}^{T} \Phi^{T} K \Phi \frac{\partial \alpha}{\partial \rho_{e}} - \sum_{i=1}^{N_{b}} \lambda_{i}^{T} \frac{\partial K_{i}}{\partial \rho_{e}} U_{i} - \sum_{i=1}^{N_{b}} \lambda_{i}^{T} K_{i} \frac{\partial U_{i}}{\partial \rho_{e}}$$
(12)

We chose the adjoint variable $\tilde{\alpha} = \alpha$ such as to eliminate derivatives of the solution vectors. Furthermore, we can define the variable of the residual vectors as expressed in Eq. 13.

$$\Delta F = F - K \Phi \alpha \tag{13}$$

Moreover by applying the equality $2\alpha^T \frac{\partial \Phi^T}{\partial \rho_e} \Delta F = \sum_{i=1}^{N_b} 2\alpha_i \frac{\partial U_i^T}{\partial \rho_e} \Delta F$ we obtain the following

simplified expression of the sensitivity:

$$\frac{\partial c}{\partial \rho_e} = -\alpha^T \Phi^T \frac{\partial K}{\partial \rho_e} \Phi \alpha - \sum_{i=1}^{N_b} \lambda_i^T \frac{\partial K_i}{\partial \rho_e} U_i - \sum_{i=1}^{N_b} \frac{\partial U_i^T}{\partial \rho_e} \left(K_i \lambda_i - 2\alpha_i \Delta F \right)$$
(14)

In order to eliminate the derivatives of the displacement vectors we define the adjoint variables λ_i as the solutions of Eq. 15.

$$K_i \lambda_i = 2\alpha_i \Delta F \qquad i = 1, \dots, N_b \tag{15}$$

Accordingly we obtain the final expression of the corrected sensitivity:

$$\frac{\partial c}{\partial \rho_e} = -\alpha^T \Phi^T \frac{\partial K}{\partial \rho_e} \Phi \alpha - \sum_{i=1}^{N_b} \lambda_i^T \frac{\partial K_i}{\partial \rho_e} U_i$$
(16)

We note that the expression of the corrected sensitivities includes two terms. The first one is the sensitivity calculated on the approximate solution. This is the only term considered in the sensitivity of the previous approach (Algorithm 1). The second term is the adjustment term, which corrects the sensitivity by taking into account that an approximate, reduced basis solution was used instead of the true solution of the problem. This correction has the potential to allow higher residuals for the approximate reduced basis solution because the sensitivity is corrected to account for this residual. In terms of implementation the algorithm of the iterative approach with corrected sensitivity can be divided in two parts. The first part (Algorithm 2, below) remains very similar to Algorithm 1 of the direct approach. The second part (Algorithm 3, below) corresponds to the new implementation of the corrected sensitivity and should be added right after the usual sensitivity calculation in a classical topology optimization code.

Algorithm 2: First part of algorithm for the iterative approach with adjusted sensitivities. The code below should replace the line " $U \leftarrow$ solution of KU = F" in a usual topology optimization algorithm.

```
1:
        switch \leftarrow 0
 2:
        if iteration = 1
 3:
              U \leftarrow \text{solution of } K U = F
 4:
              Store factorized matrix K
 5:
              Add U/\operatorname{norm}(U) to matrix \Phi
 6:
        else if 1 < iteration \le N_b
 7:
              U \leftarrow \text{solution of } K U = F
              Store factorized matrix K
 8:
              U_{ortho} \leftarrow U - \Phi (\Phi^T U)
 9:
10:
               Add U_{ortho} / norm(U_{ortho}) to matrix \Phi
11:
         else
               \alpha \leftarrow \text{solution of } (\Phi^T K \Phi) \alpha = \Phi^T F
12:
              e_{rh} \leftarrow \operatorname{norm}(K \Phi \alpha - F) / \operatorname{norm}(F)
13:
              if e_{rb} > \varepsilon
14:
                      Remove oldest basis vector from matrix \Phi
15:
                       U \leftarrow \text{solution of } K U = F
16:
17:
                        Store factorized matrix K
                       U_{ortho} \leftarrow U - \Phi (\Phi^T U)
18:
                      Add U_{ortho} / \operatorname{norm}(U_{ortho}) to matrix \Phi
19:
20:
              else
                       U \leftarrow \Phi \alpha
21:
22:
                      switch \leftarrow 1
23:
              end if
24:
         end if
```

Algorithm 2 is very close to Algorithm 1 proposed in the direct approach. The only changes are a logical switch which is equal to 0 if the full solution of the equilibrium equations is used and equal to 1 if only the reduced basis solution is used (line 22). This switch is required since the expression of the sensitivity of the objective function is calculated differently in each case. Furthermore the stiffness matrices corresponding to the displacement vector generating the current reduced basis subspace are stored in their factorized form (lines 4, 8 and 17) since they are reused later for calculating the adjoint variables.

Algorithm 3 is implementing the corrected sensitivity of Eq. 16 in cases where the reduced basis solution was used (line 1). Note that this algorithm is not intended to replace the line calculating the sensitivity in a usual topology optimization code. Instead it should be placed right after this line, since it complements the classical sensitivity expression. On line 3 the adjoint variable is calculated. Finding this solution involves the inversion of a system of equations of size n, note however that the stiffness matrix K_i is already available under its factorized form, as it was stored during Algorithm 2. Line 5 corrects the expression of the sensitivity usually calculated.

Algorithm 3: Second part of the algorithm for the iterative approach with adjusted sensitivities. The code below should be placed right after the line calculating the non-corrected sensitivity $\frac{\partial c}{\partial \rho_e}$ in a usual topology optimization

algo	rithm.
1:	if switch = 1
2:	for <i>i</i> from 1 to N_b
3:	$\lambda_i \leftarrow \text{solution of } K_i \lambda_i = 2\alpha_i \Delta F \text{ using already factorized } K_i$
4:	end for
5:	$\frac{\partial c}{\partial \rho_e} \leftarrow \frac{\partial c}{\partial \rho_e} - \sum_{i=1}^{N_b} \lambda_i^T \frac{\partial K_i}{\partial \rho_e} U_i$
6:	end if

Compared to the direct iterative approach, this approach has the advantage of accounting for the approximation error when calculating the sensitivities. This means that fewer full simulations are likely to be required. On the downside this approach is more computationally expensive since it requires to calculate the adjoint variables. Even though this additional cost is decreased by storing the factorized stiffness matrices, required for the adjoint variables calculation, the number of adjoint variables calculations is large (N_b times the number of reduced basis iterations). This means that the approach can only be efficient for very large systems for which the cost of factorizing the stiffness matrix is large compared to the other costs, including that of backsubstitution, given the factorized stiffness matrix.

4 Numerical investigations

4.1 Description of the topology optimization problem

To investigate the effectiveness of the two approaches presented, we consider here a classical minimum compliance benchmark problem: a beam in both 2D and 3D. The boundary conditions of the beam for the 2D case are represented in Figure 2 (this is a classical MBB beam problem). Note that only half of the structure was modeled using symmetry conditions on the left side. We thus seek the optimal material distribution inside the beam for minimum compliance given an upper bound on the material volume fraction.



Figure 2: Boundary conditions for the 2D MBB beam problem

The general parameters considered for the topology optimization problem were: nominal and minimum Young's modulus $E_{nominal} = 1$ and $E_{min} = 1*10^{-9}$, Poisson's ratio v = 0.3. For the 2D problem we considered a maximum allowable volume fraction of 0.5, a penalization factor p = 3 and a density filter radius of 1.5, while for the 3D problem the maximum allowable volume fraction was 0.3 (the other parameters remaining unchanged compared to the 2D problem). The optimization iterations are stopped either when the density variation within any of the elements is less than 1% or when the total number of iterations has reached the maximum number of allowable iterations, whichever criterion is met first.

The code implementation used was adapted from Andreassen et al. [25] for the 2D topology optimization problem and from Liu and Tovar [26] for the 3D problem.

4.2 Results using the direct iterative approach

Several test problems were used to compare the performance of the proposed approach. Both a low density and a high density mesh were first considered for the 2D MBB beam problem. Then a 3D problem was also investigated.

Table 1 presents the results of the topology optimization for the 2D case with a low density mesh of 150 x 50 elements. The first line of the table provides the reference solution obtained by running the topology optimization without the reduced order models. The next lines provide the results of the topology optimization with the direct iterative reduced basis approach for different values of the user defined parameters ε (residuals threshold) and N_b (size of the reduced basis). The third and fourth columns provide the time speedup of the proposed approach, i.e. the CPU time of the reference topology optimization solution divided by the CPU time of the proposed approach. The third column provides the speedup when including the stiffness matrix assembly time, while the fourth column provides this speedup when not including it. The reason why both speedups are provided is because the time for assembling the stiffness matrix can be quite significant, especially for small and medium size problems. However the process of assembling the stiffness matrix is straight forward to parallelize on multiple CPUs (with a parallel for loop), which can thus quickly reduce the assembly time to negligible values. The fifth column provides the total number of iterations in the topology optimization routine. This is either the number of iterations until the convergence criterion was met or until the total number of available iterations (400 here) was reached. Note that for the considered problem it was always the maximum number of iterations that was reached. The sixth column provides the number of full solutions of the equilibrium equations that had to be computed throughout the optimization. Subtracting this value from the total number of iterations thus provides the number of times that only the reduced basis solution was used. Finally, the last column provides the relative error in the objective function at the optimum for the approximate solution obtained using the reduced order models, compared to the reference solution. This error (given in %) is expressed in Eq. 17.

$$err = \frac{c_{approx} - c_{exact}}{c_{exact}} \times 100$$
(17)

Where *c* represents the value of the objective function (compliance) at the optimal design.

Residuals threshold, ε	Size of reduced basis, N _b	Time speedup with assembly	Time speedup without assembly	Number of iterations	Number of full solutions	Error in objective function (%)
-	-	1	1	400	400	0
0.1	4	1.26	1.58	400	116	-0.022
0.1	10	1.20	1.45	400	117	-0.021
0.1	40	1.03	1.07	400	106	-0.011
0.05	4	1.23	1.49	400	142	-0.010
0.05	10	1.17	1.36	400	142	-0.010
0.05	40	1.01	1.03	400	128	-0.012
0.01	4	1.05	1.10	400	263	-0.004
0.01	10	1.00	1.01	400	263	-0.004
0.01	40	0.93	0.90	400	190	-0.003

Table 1. Relative computational cost associated to the direct iterative approach for different parameters for a 150 x 50 mesh density.

Figure 3 provides the obtained topologies for the reference routine without reduced order modeling as well as for the proposed direct iterative approach with reduced order modeling using the middle values of the user defined parameters: $\varepsilon = 5*10^{-2}$ and $N_b = 10$. The images of Figure 3 are direct plots of the density value of each pixel, without any other post-processing. Note that for all other values of the user defined parameters the results were the same, i.e. the difference between the approximate solution and the reference solution were undistinguishable to the naked eye.



Figure 3. Optimal topologies with a 150x50 mesh size for : a) the reference topology optimization routine b) the proposed direct iterative approach with $\varepsilon = 5*10^{-5}$ and $N_b = 10$

From Table 1 we can conclude that relatively small values for the reduced basis size N_b are already sufficient to obtain good solutions. This size has very little impact on the error in the final value of the objective function. On the other hand it has a clear impact on computational cost, with larger bases requiring more time. Clearly a size of 40 is overkill for this problem and also turns out to be inefficient since the "speedup" obtained is lower than 1. Here a size of 4 appears enough to obtain good accuracy and topology optimization results that are indistinguishable to the naked eye from the reference solution. It is important to realize what a reduced basis size of 4 exactly implies: for the problem on the second line of Table 1 this

means that for 284 out of 400 simulations done during the optimization, it was enough to compute the solutions projected on a basis of only four displacement vectors.

Table 1 also indicates that the residuals threshold ε is a critical parameter since it significantly affects both the efficiency and the accuracy of the approach. While the accuracy gains are minor here (the accuracy of the final solution is acceptable even for the highest threshold) the efficiency improvement for high residual thresholds is non negligible.

Note also that the efficiency improvement is quite modest here (maximum speedup of 1.58) but this is due to the problem being really small. Much higher speedups are achievable for larger scale problems as will be shown a bit further down on the 3D test problems.

In order to have a better understanding of the reduced basis behavior it is interesting to analyze the evolution of the residuals throughout the optimization (Figure 4). A similar trend can be observed in all 6 curves. At the beginning of the optimization the residuals are relatively high, implying that the full solutions of the equilibrium equations need to be calculated. At some point however, the residuals go below the residuals threshold (red horizontal line in Figure 4), meaning that for this and the following iterations for which the residuals are below the threshold only the reduced basis solution is sufficient. From time to time the residuals exceed once again the threshold, thus requesting a new full solution, which will be used to update the reduced basis and thus guarantee that the reduced basis solutions will again be acceptable over a few iterations.

When comparing the first three plots of Figure 4, where only the size of the reduced basis was varied we can see that the trends are very similar among the three curves. The main difference is that the curves start at later iterations for high values of N_b , since for the first N_b iterations the full solution is always computed in order to construct the initial reduced basis.

Note also that the regular update of the reduced basis (each time the residuals exceed the threshold) also assures that the vectors of the reduced basis are current with respect to the latest optimization step (if the reduced basis wasn't updated and the initial basis vectors were used all along the optimization the results would be unacceptably poor).





Figure 4. Plot of the residuals values with the optimization iteration for different values of the user defined parameters for a 150×50 mesh.

A similar investigation was carried out using a high density mesh of 600 x 200 elements and the results are provided in Table 2 and Figure 5. The topologies in Figure 5 exhibit some very small scale topological details, which allow observing small differences between the various solutions. Note that if these fine topological details are not desired, the filter radius could be increased to avoid them.

Table 2. Relative computational cost associated to the direct iterative approach for different parameters for a 600 x 200 mesh.

Residuals threshold, ε	Size of reduced basis, N _b	Time speedup with assembly	Time speedup without assembly	Number of iterations	Number of full solutions	Error in objective function (%)
-	-	1	1	400	400	0
0.1	4	1.47	1.80	400	149	-0.046
0.1	10	1.38	1.63	400	152	-0.054
0.05	4	1.33	1.55	400	188	-0.012
0.05	10	1.29	1.47	400	188	-0.010





Figure 5. Optimal topologies with a 600x200 mesh size for : a) the reference topology optimization routine b) the proposed direct iterative approach with $\varepsilon = 0.1$ and $N_b = 4$ c) $\varepsilon = 0.1$ and $N_b = 10$ d) $\varepsilon = 0.05$ and $N_b = 4$

Based on the numerical results obtained with various combinations of the user defined parameters N_b and ε we can make following comments regarding the choice of these parameters.

When analyzing the effect of the reduced basis size N_b we can again note that it has some impact on the efficiency but barely any on the accuracy of the final result. Accordingly it appears advisable to always start with using a low reduced basis size. Going much lower than 4 is not recommended since the efficiency gains would be negligible but for some problems the small size could be problematic. Note that a too small reduced basis (i.e. size of one or two) may even be counterproductive even in terms of efficiency since the residual threshold will be exceeded more often with such a small reduced basis, thus requiring more full simulations. A reduced basis size of two is to be considered an absolute minimum for this direct iterative approach.

At the other end, a reduced basis size above 10 seems to be overkill and significantly degrades the efficiency of the problem. In the author's experience a reduced basis size between 2 and 10 would work well for the vast majority of minimum compliance topology optimization problems. A size of four seems a safe starting point.

The residuals threshold ε appears again to be the more critical of the two parameters since it has a significant effect on both the accuracy and the efficiency of the approach. For the high value of the threshold ($\varepsilon = 0.1$) we can observe some discrepancies in the approximate optimal topologies of Figure 5 b) and c) and the reference optimal topology of Figure 5 a). The differences in some of the details of the topology are due to the residual threshold being too high, not allowing the reduced basis approximation to model the area around these fine topology details accurately enough. We can note that for the lower value of the threshold ($\varepsilon = 0.05$) the approximate optimal topology is undistinguishable to the naked eye from the reference optimal topology (only the optimum for $N_b = 4$ is represented in Figure 5.d but the optimum for $N_b=10$ is the same to the naked eye).

Choosing the appropriate value of the residual threshold for a given problem appears to be somewhat more difficult than choosing the basis size, because the threshold is much more problem dependent. A possible way to determine an acceptable threshold value for large scale problems is to proceed in a sequential way using multiple mesh sizes. This is similar to the idea behind several of the multiresolution and multimesh approaches reviewed in [3]. The topology optimization would be carried out using the full and the reduced basis approach on a coarse mesh for multiple values of the residuals threshold. Note that the basis size N_b can also be varied but it was found to have relatively little effect (see Tables 1 and 2 and previous comments). This procedure allows to determine the most appropriate value of the parameter for this coarse mesh. Then when moving to the full scale problem with a high density mesh this initial value of the threshold would be divided by a knockdown factor depending on i) the accuracy in the topology details one is looking for ii) the magnitude of the difference between the low density and the high density mesh iii) the available computational resources.

Note that such a sequential multimesh approach can be easily automated and the mesh density of the coarse mesh can be chosen such as to keep computational cost small compared to the expected gains of using the proposed reduced basis method. Of course the analysis of the results and the final choice of the parameters would have to be done by the user. However, such a sequential approach is not found to be fundamentally disruptive of traditional practice, since users often experiment with the behavior of the problem on coarse meshes before launching the run with the full scale mesh density, which may take several days.

Figure 6 illustrates the evolution of the residuals with the optimization iterations for the 600 x 200 mesh. The same overall trends are found as for the 150 x 50 mesh and the same comments apply.



Figure 6. Plot of the residuals values with the optimization iteration for different values of the user defined parameters for a 600 x 200 mesh.

Since 3D topology optimization problems are usually larger and the inversion of the equilibrium equations is usually computationally more intensive, the results for a 3D problem with a 25 x 25 x 14 mesh has also been investigated and the results are presented in Table 3 and Figure 7. Note that similarly to the images of the 2D topologies, the images in Figure 7 represent the 3D plots of the density values of each voxel. In order to let the inner structure be visible as well, voxels with a density value below a certain threshold (0.25 here) are not displayed.

parameters	Auditeter for a 20 A 20 A 1 + moon density.							
Residuals	Size of	Time	Time speedup	Number	Number	Error in		
threshold,	reduced	speedup with	without	of	of full	objective		
З	basis, N_b	assembly	assembly	iterations	solutions	function (%)		
-	-	1	1	200	200	0		
0.1	4	3.30	4.74	200	38	-0.030		
0.1	10	3.26	4.64	200	37	-0.063		
0.05	4	2.82	3.72	200	48	-0.020		
0.05	10	3.08	4.26	200	41	-0.027		

Table 3. Relative computational cost associated to the direct iterative approach for different parameters for a 25 x 25 x 14 mesh density.



Figure 7. Optimal topologies with a 25 x 25 x 14 mesh size for : a) the reference topology optimization routine for a 3D cantilever beam problem b) the proposed direct iterative approach with $\varepsilon = 0.1$ and $N_b = 4$ c) $\varepsilon = 0.1$ and $N_b = 10$ d) $\varepsilon = 0.05$ and $N_b = 4$

The effect of the size of the reduced basis is confirmed again in this study. Note that the efficiency difference between a reduced basis size of 4 and 10 is much lower here for the large scale 3D problem compared to the previous medium scale 2D problem. Accordingly, the choice of the basis size becomes less and less critical with the size of the problem.

In terms of the optimal topologies found with different parameter combinations there are small differences in terms of element densities, visible to the naked eye when comparing to the reference solution. Nevertheless all topologies are very close to each other in terms of the predominant attributes.

The time speedup achieved for this problem is quite significant, reaching 3.3 when considering the stiffness matrix assembly and even 4.74 when parallelizing the assembly. This is achieved by the proposed approach by computing the full solution for only 37 out of the 200 iterations in the best case (third line of Table 3). At the 163 other iterations only the reduced basis solution is computed.

The evolution of the residuals along the optimization is presented in Figure 8. A similar behavior is found here as for the 2D problem with the low density mesh (Figure 4). Note that the evolution for the 2D problem in Figure 4 may appear more noisy/irregular but this is partly due to the total iteration numbers being different (400 vs 200 total iterations). When zooming on Figure 4 we can observe very similar behavior to the ones here.



Figure 8. Plot of the residuals values with the optimization iteration of the direct iterative approach for different values of the user defined parameters for a $25 \times 25 \times 14$ mesh.

4.3 Results using the iterative approach with adjusted sensitivities

The iterative approach with adjusted sensitivities is only efficient for large scale problems as explained in section 3.2. Accordingly only the 3D problem was considered here. Furthermore, the approach cannot be efficient with a large basis size, so N_b was selected to vary only between 1 and 4. The results of the topology optimization implementing this approach are presented in Table 4.

Residuals threshold, ε	Size of reduced basis, N _b	Time speedup with assembly	Time speedup without assembly	Number of iterations	Number of full solutions	Error in objective function (%)
-	-	1	1	200	200	0
0.1	1	2.34	2.80	200	23	0.30
0.1	2	1.75	1.94	200	23	0.30
0.1	3	1.39	1.47	200	23	0.30
0.1	4	1.12	1.14	200	23	0.29
0.05	1	2.09	2.44	200	32	0.32
0.05	2	1.63	1.78	200	32	0.32
0.05	3	1.30	1.35	200	32	0.33
0.05	4	1.07	1.08	200	32	0.33

Table 4. Relative computational cost associated to the iterative approach with adjusted sensitivity for different parameters for a $25 \times 25 \times 14$ mesh density.

The corresponding optimal topologies are presented in Figure 9 for the two cases with the highest error in the objective function for each of the two residuals threshold values. Note that the reference solution can be seen in Figure 7.a).



Figure 9. Optimal topologies with a 25 x 25 x 14 mesh size the proposed iterative approach with adjusted sensitivities with a) $\varepsilon = 0.1$ and $N_b = 1$ b) $\varepsilon = 0.05$ and $N_b = 3$

Several interesting remarks can be made when analyzing Table 4 and Figure 9. First of all the error in the objective function is higher than that of the direct iterative approach. While the error is still small and the optimal topologies in Figure 9 are still visually very close to the reference solution it is important to note this discrepancy. On the other hand we are able to push the user defined parameters to much more extreme values, which would be unacceptable for the direct iterative approach. Indeed even with a basis reduced to the minimum ($N_b=1$) and a high residual threshold ($\varepsilon=0.1$) we still obtain acceptable results (roughly the same accuracy as with stricter values for these parameters). This is due to the sensitivity correction by the adjoint variables, which allows higher residuals and lower dimensional reduced bases, without accuracy penalty. Obviously the highest speed-up is also obtained for these parameters. It appears that a basis size of more than four is to be disadvised, since no speedup will be achievable. This is due to the large number of adjoint variable calculations required for the sensitivity correction for large basis sizes.

Finally it is interesting to compare the residuals convergence (Figure 8) to that of the direct iterative approach. We look here at the two extreme cases in terms of the user defined parameters.



Figure 10. Plot of the residuals values with the optimization iteration of the iterative approach with adjusted sensitivities for different values of the user defined parameters for a 25 x 25 x 14 mesh.

When comparing the evolution of the objective function for the proposed approach with adjusted sensitivities (Figure 10) to the direct iterative approach (Figure 6) we can note that the residual value stays over a much larger number of iterations below the threshold between two full simulations. This is obviously the effect of the sensitivity adjustement through the adjoint variables.

4.4 Scaling of the computational savings on the 3D beam problem

In order to further investigate how the computational savings of the proposed methods scale up with the size of the problems we run the topology optimization for two additional test cases. We considered the previous 3D beam topology optimization problem with a 50x50x28 mesh in this subsection and a 3D wheel problem with a 148x148x74 mesh in the next subsection.

According to previous comments the direct iterative approach was run with a reduced basis of $N_b = 4$ and a residuals threshold of $\varepsilon = 0.1$. Similarly the iterative approach with adjusted sensitivities was run with a reduced basis of $N_b = 1$ and the same residuals threshold of $\varepsilon =$

0.1. The optimal topologies obtained with each approach are shown in Figure 11 and the evolution of the objective function in Figure 12. The numerical results are summarized in Table 5.





Figure 12. Plot of the residuals values with the optimization iteration for a 50 x 50 x 28 mesh.

Approach	Time speedup with assembly	Time speedup without assembly	Number of iterations	Number of full solutions	Error in objective function (%)
Reference topology optimization	1	1	200	200	-
Direct iterative approach $(N_b=4, \epsilon=0.1)$	3.47	4.84	200	35	-0.029
Iterative approach with adjusted sensitivities $(N_b=1, \ \varepsilon=0.1)$	2.77	2.82	200	19	0.795

Table 5. Numerical results for the 3D problem with a 50 x 50 x 28 mesh density

Analyzing these results we can see that the computational savings continue to grow with the size of the problem. The direct iterative approach runs 35 full simulations out of 200 total runs. This means that the maximum achievable speedup with these runs would be of 5.71. Without assembly time (i.e. parallelized assembly) the actually achieved speedup is 4.84, meaning that 0.87 represents the overhead cost of the reduced basis method. Note that with increased size of the problem this overhead cost has gone steadily down. When further increasing the mesh density the speedup converges towards the maximum achievable one for the considered problem, as will be illustrated by the problem in the next subsection.

For the iterative approach with adjusted sensitivities we have 19 full simulations out of the 200 total runs. This means that the maximum achievable speedup is 10.5 here. The actually achieved speedup even without assembly time (i.e. parallelized assembly) is 2.82. This means that the overhead cost of the iterative approach with adjusted sensitivities is much higher. This is to be expected though, since even though the matrices are stored in factorized form the back-substitution required for adjusting the sensitivities still requires significant time and is repeated a large number of times. Note though that the overhead cost of this approach has also been steadily declining with the size of the problem. Since the maximum achievable speedup is larger for this method than for the direct iterative method it is thus possible that for very large scale problems this method will be the more efficient one between the two. Note however that the error in the iterative approach with adjusted sensitivities is much higher than for the direct iterative approach with adjusted sensitivities is much higher than for the direct iterative approach with adjusted sensitivities is much higher than for the direct iterative approach. This is most probably due to the very limited reduced basis $(N_b = 1)$.

When comparing the direct iterative approach to the approach with adjusted sensitivities one can make some additional comments. The comparison of the speedups on the application problems shows that the direct approach is able to achieve a greater speedup, while also providing higher accuracy (lower objective function error). For very large problems the approach with adjusted sensitivities may start have an efficiency advantage over the direct approach due to the reduced number of full simulations that it requires. Note that this advantage of the approach with adjusted sensitivities can only happen with the size of the reduced basis shrinked to the minimum ($N_b = 1$) in order to have as few adjoint variables calculations as possible. Obviously such a low dimensional reduced basis may pose some issues in terms of accuracy for some problems. Note also that the efficiency of the approach with adjusted sensitivities depends largely on the efficiency of the factorization compared to the back-substitution; indeed for calculating the adjoint variables the back-substitution still needs to be carried out even if the factorized matrix is already available. The ratio of the time required for factorization over the time required for back-substitution is problem dependent, being notably affected by the sparsity structure of the initial matrix. This means that the

efficiency gains of the approach with adjusted sensitivities may somewhat vary depending on the problem's structure. They will also depend on the efficiency of the factorization and backsubstitution algorithms employed, with the Matlab algorithms being used here not necessarily being the most efficient.

Overall the direct iterative approach appears to be a relatively straight forward approach with a potential for already very significant computational savings (by up to a factor of 4.8 and the efficiency gains have been shown to increase with the size of the problem). While the approach with adjusted sensitivities has at least in theory the potential to exhibit even greater computational savings for very large problems it also appears to be garnered with more pitfalls, so special attention will need to be taken when implementing it on a specific problem. Finally let us note that the approach with adjusted sensitivities has much higher RAM memory requirements than the direct iterative approach. Indeed for calculating the adjoint variables, the factorized stiffness matrices of every simulation need to be explicitly calculated and stored which can be quite memory intensive. Since the direct iterative approach has lower memory requirements a further test case is provided in the next subsection on a problem with over a million degrees of freedom.

4.5 Scaling of the computational savings on a 3D wheel problem

An additional, more complex 3D problem, known as the 3D wheel problem, is considered in this section to further investigate the scaling of the computational cost savings for the direct iterative approach. Figure 13 illustrates the boundary conditions of this problem. The solid is simply supported at the four corners and a downward force is applied at the center of its bottom face. A mesh of 148x148x74 is considered in this test case. Using symmetry boundary conditions only one quarter of the volume is modeled, which accounts for about 1.26 million degrees of freedom in the finite element model.



Figure 13: Boundary conditions for the 3D wheel problem

The results of the reference topology optimization approach and of the direct iterative approach with a reduced basis size of $N_b = 4$ and a residuals threshold of $\varepsilon = 0.015$ are presented in Figure 14. The numerical results are summarized in Table 6, while Figure 15

provides the evolution of the residuals throughout the optimization for the direct iterative approach.



Figure 14. Optimal topologies for the 3D wheel problem with a 148 x 148 x 74 mesh size for : a) the reference topology optimization routine b) the direct iterative approach with $\varepsilon = 0.015$ and $N_b = 4$



Figure 15. Plot of the residuals value with the optimization iteration for the 3D wheel problem with a 148 x 148 x 74 mesh.

Approach	Time speedup with assembly	Time speedup without assembly	Number of iterations	Number of full solutions	Error in objective function (%)
Reference topology optimization	1	1	300	300	-
Direct iterative approach $(N_b=4, \epsilon=0.015)$	12.1	12.7	300	25	-0.032

Table 6. Numerical results for the 3D problem with a 148 x 148 x 74 mesh

The results obtained for this 3D problem with about 1.2 million degrees of freedom in the finite element analysis further demonstrate the computational cost savings potential of the direct iterative approach. A maximum speedup by a factor of 12.7 was achieved. Note that for such large scale problems the cost of solving the system of equations of Eq. 3 is by far the dominant computational cost, accounting for about 95% of the computational time. Both the stiffness matrix assembly and the other overhead costs of the method become negligible for this 3D problem with over a million degrees of freedom. Note also that the potential gains that could be obtained by parallelizing the stiffness matrix assembly (second vs. third row of Table 6) are relativly small given the previous remark.

Before concluding let us comment on how the computational savings of the two methods proposed compare to those of other reduced basis topology optimization methods. Wang et al. [12], who used Krylov subspaces, reported savings by a factor of 3.6 for 107 184 design variables. Amir et al. [14], who used the combined approximations approach to construct a reduced basis, reported savings by up to a factor of 5.66 for 36 864 design variables. The approaches proposed here thus allow to achieve similar or higher savings, while at the same time being relatively simple and straightforward to implement in already existing topology optimization frameworks.

Of course other methods, not involving reduced basis modeling, have been proposed for reducing the computational cost of topology optimization. The methods proposed in this work have however the potential to be combined in a synergetic way with many of these other methods. For example advances in the numerical solvers of systems of equations can be combined with the proposed reduced basis approach to achieve even greater savings. The methods described should thus not be seen as competitors to many other topology optimization approaches but rather as possible candidates for further synergies.

5 Conclusions

A new method for efficiently solving large scale minimum compliance topology optimization problems has been proposed. The approach is based on the coupling of topology optimization with reduced order modeling by projection (or reduced basis modeling). A new approach for constructing the reduced basis *on the fly*, using previously calculated solutions of the equilibrium equations is proposed.

A direct iterative approach was presented as well as an approach with adjusted sensitivities, which takes into account the residual error when calculating the sensitivity of the objective function.

The direct iterative approach is very straight forward to implement and showed large speedups over traditional topology optimization without reduced order modeling. Speed-ups up to a factor of 12 were achieved and this speedup was shown to increase with the size of the problem: the larger the problem and the more time consuming a single solution of the equilibrium equations will be, the higher the efficiency gains allowed by calculating the full solutions at only some iterations while using quick reduced order models at the other.

The iterative approach with adjusted sensitivities appears to have more implementation pitfalls but has at least in theory the potential to lead to even larger speedups for very large problems.

Finally, note that the reduced basis construction approach was proposed here for minimum compliance structural topology optimization problems. However the proposed *on the fly* reduced basis construction is very generic and could potentially be applied to other topology optimization approaches, whether in the structural or other scientific domains (e.g. thermal analysis, electromagnetics, etc). It is worth noting that the major condition for the proposed reduced basis topology optimization methods to be applicable to other fields is that the equilibrium equations can be discretized such as to lead to a linear system of equations that needs to be solved to obtain the solution. If this is the case, the system of equations can be solved projected on a reduced basis and the proposed on the fly reduced basis enrichment approach can be directly adapted. Investigating this applicability to other areas and other frameworks is an interesting outlook to be explored.

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