Optimization Based Algorithms for Uncertainty Propagation Through Functions with Multidimensional Output Within Evidence Theory

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Evidence theory is one of the approaches designed specifically for dealing with epistemic uncertainty. This type of uncertainty modeling is often useful at preliminary design stages where the uncertainty related to lack of knowledge is the highest. While multiple approaches for propagating epistemic uncertainty through one-dimensional functions have been proposed, propagation through functions having a multidimensional output that need to be considered at once received less attention. Such propagation is particularly important when the multiple function outputs are not independent which frequently occurs in real world problems. The present paper proposes an approach for calculating belief and plausibility measures by uncertainty propagation through functions with multidimensional, non-independent output by formulating the problem as one-dimensional optimization problems in spite of the multidimensionality of the output. A general formulation is first presented followed by two special cases where the multidimensional function is convex and where it is linear over each focal element. An analytical example first illustrates the importance of considering all the function outputs at once when these are not independent. Then an application example to preliminary design of a propeller aircraft then illustrates the proposed algorithm for a convex function. An approximate solution, found to be almost identical to the exact solution is also obtained for this problem by linearizing the previous convex function over each focal element.

Keywords: evidence theory, uncertainty propagation, optimization, non-independence, preliminary aircraft design

I. Introduction

The engineering community has increasingly recognized over the past few decades the need for accounting for uncertainties in design, optimization and risk assessment of complex systems such as aircraft, spacecraft or nuclear power plants [1][2]. Uncertainty can manifest itself under multiple forms and a distinction is usually made between aleatory uncertainty and epistemic uncertainty. Aleatory uncertainty is also known as stochastic uncertainty,

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irreducible uncertainty, inherent uncertainty, variability or type I uncertainty. It can stem from environmental stochasticity, inhomogeneity of materials, fluctuations in time, variation in space, heterogeneity or other intrinsic differences in the features of a system. Epistemic uncertainty on the other hand, sometimes called reducible uncertainty or type II uncertainty, stems from lack of knowledge. This kind of uncertainty can be related to scientific ignorance, measurement uncertainty (e.g. sensor uncertainty), insufficient experimental data, phenomena inobservability or censoring, thus in general terms lack of knowledge. Note that the distinction between aleatory and epistemic uncertainty is not always straightforward. Measurement uncertainty for example can be treated as aleatory uncertainty if enough data is available to fit it with a probability distribution. However if such information is not available, then measurement uncertainty can be treated as epistemic uncertainty. Finally, some authors define error as a third type of uncertainty. For detailed discussions on the different types of uncertainty the reader can refer to [3],[4].

The best known and most widely used method for representing uncertainty is probability theory, which is particularly well suited for representing aleatory uncertainty. While it can be also used for describing epistemic uncertainties, especially under the generalized theory of imprecise probabilities [5],[6], there has been some debate [7],[8] whether there are better ways to represent epistemic uncertainty. Evidence theory is one of the alternative theories proposed for treating epistemic uncertainty and is the theory on which we focus on in the present article.

A critical question in evidence theory, as in most uncertainty representation theories, is the following: how to propagate uncertainty through a function in an economical way. For functions with one-dimensional output, the determination of the belief and plausibility subsets can be easily formulated in terms of single criterion optimization problems. Propagation of uncertainty within evidence theory through functions with vector outputs (i.e. multi-dimensional functions) has however obtained much less attention in spite of the fact that such cases may often arise in real world problems. This is in part because the methods for one-dimensional cases can still be applied to multidimensional ones, as illustrated in [9], if the outputs are independent (i.e. non-interactive) or close to being so. We refer the reader to [10],[18] for an overview of the various concepts of independence in evidence theory. In order to account for the output non-independence in uncertainty propagation, new formulations need to be considered, since the one-dimensional optimization formulations trivially available for functions with one-dimensional output are no longer directly applicable due to the absence of a total order over IR^n.

The aim of the present article is to propose an approach for evidence uncertainty propagation through functions with multidimensional, non-independent output by formulating it in terms of a one-dimensional optimization problem in spite of the multidimensionality of the output. A general formulation and the corresponding implementation algorithms are first presented. Specific formulations are then derived in two particular cases: when the function is convex and when it is linear over each focal element. These specific formulations benefit from the availability of efficient dedicated one-dimensional optimization algorithms. Two application problems are considered: one illustrative analytical example and an application problem to aircraft preliminary sizing.

The rest of the article is organized as follows. Section II presents an overview of evidence theory. In Section III we present the proposed approach for calculating the belief and plausibility measures by uncertainty propagation through functions multidimensional functions with non-independent output. This approach is based on formulating the calculation of the belief and plausibility subsets in terms of one-dimensional optimization problems. Section IV gives an analytical illustration example while Section V provides an application to a preliminary aircraft design problem. Finally, section VI provides concluding remarks.

II. Evidence theory

Evidence theory is a theory initially developed by Dempster [14] and Shafer [15] (the theory is also known as Dempster-Shafer theory), and aimed specifically at improving the representation of epistemic uncertainty.

From a formal point of view, evidence theory is applied on an evidence space *ES*:

$$ES = \{US, S, m\} \tag{1}$$

where *US* is a set that contains everything that could occur in the universe under consideration, *S* is a usually countable collection of subsets of *US* and *m* is a function defined on the powerset of *US* such that:

$$m(\Gamma) = \begin{cases} > 0 & \text{if } \Gamma \in S \\ 0 & \text{if } \Gamma \subset US, \Gamma \notin S \end{cases} \quad \text{and} \quad \sum_S m = 1 \,. \tag{2}$$

For a given collection of subsets *S*, *m* is a number characterizing the amount of likelihood that can be assigned to each subset. Using the terminology of evidence theory a subset Ω_i for which *m* is different from zero is called a focal element. We then have $m_i = m(\Omega_i)$ denominated the basic probability assignment (BPA) associated to a focal element.

It is important to note that m_i is not an uncertainty measure itself. Instead, in evidence theory these are plausibility (denoted *Pl*) and belief (denoted *Bel*) of an event $e \subset US$, defined as follows:

$$Pl(e) = \sum_{\Omega_i \cap e \neq \varnothing} m(\Omega_i) \tag{3}$$

$$Bel(e) = \sum_{\Omega_i \subset e} m(\Omega_i) \tag{4}$$

From a conceptual point of view, $m(\Omega_i)$ can be considered to be the amount of likelihood that is associated with Ω_i, but without specifying how this likelihood can be distributed within Ω_i. Accordingly, the belief *Bel(e)* can be viewed as the minimum amount of likelihood that must be associated with *e* and the plausibility *Pl(e)* can be viewed as the maximum amount of likelihood that could be associated with *e*.

The collection of subsets *S* can be defined such as to exclude scattered, nested or overlapping subsets. A collection of adjacent polytopes (i.e. bounded polyhedrons) verifies these conditions and is typically used in evidence theory. We thus also consider a polytopical Dempster Shafer structure (cf. Figure 1) in the present article.

To illustrate the concept of belief and plausibility we assume that the universal set is partitioned into nine subsets Ω_1 to Ω_9 and each subset Ω_i is accorded a BPA value, m_i, as shown in Figure 1. Let us assume we seek the belief and plausibility of the event *e*, which is given by the shaded area in Figure 1. According to the previous definitions the degree of plausibility *Pl(e)* is calculated by adding up the BPA of the subsets whose intersection with *e* are not an empty set. That is, every proposition that allows *e* to be included at least partially is considered to imply the plausibility of *e*. *Bel(e)* is calculated by summation only of the BPA of the subsets that are included in *e*. On the example of Figure 1, *Bel(e)* is obtained by adding up the BPA of Ω_3 and Ω_6 that are totally included in the shaded area. *Pl(e)* is obtained by adding up the BPA of Ω_2, Ω_3, Ω_5, Ω_6 and Ω_9 which are partially or totally included in *e*. Belief and plausibility can then be seen as lower, respectively upper bounds of an unknown probability.

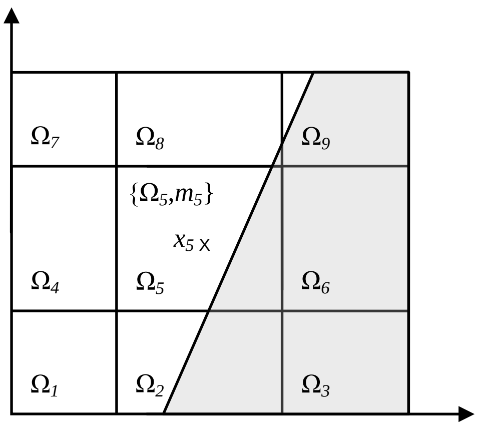

Figure 1. Example of Dempster-Shafer subsets structure. The shaded area represents event *e*.

The previous definitions and properties lead to the following uncertainty representation in evidence theory (see Figure 2). Note that the uncertainty is defined here as the difference between plausibility and belief of an event *e*. In terms of subsets this can be also seen as everything that is neither totally included in *e* nor in its complementary.

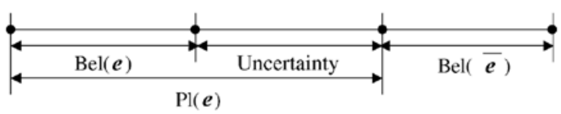

Figure 2. Belief and plausibility representation in evidence theory.

As in other uncertainty representation theories one of the major hurdles lies in the propagation of uncertainty through a function, which usually models some physical phenomena. The function may be quite complex and may be known only at specific points through the use of a black-box simulator (e.g. finite element simulation). Both the input and the output of the function can be multi-dimensional vectors.

The focus of this paper is on uncertainty propagation through functions with multidimensional output vectors. The multi-dimensional output can pose a problem and prevent a component by component analysis whenever the different output components are not independent, which would usually be due to the nature of the considered function. In the next sections we propose several algorithms for calculating belief and plausibility for functions with multidimensional output. We distinguish several cases depending on the nature of the function or of its approximations. We present first the general formulation of the belief and plausibility functions for multidimensional cases, then several methods in particular cases aiming at reducing the computational burden.

III. Uncertainty propagation approaches for belief and plausibility

A. General problem formulation

Let us consider a function g defined as:

$$\begin{aligned} g : \mathbb{R}^n &\to \mathbb{R}^p \\ x &\mapsto y = g(x) \end{aligned} \tag{5}$$

Uncertainty is considered in the function's input parameters x and it is modeled within the evidence theory framework. A basic probability assignment (BPA) is assigned to a total of k focal elements, characterized by the couple $\{\Omega_i, m_i\}$, where Ω_i is a subset of $\mathbb{R}^n$ and m_i the corresponding BPA. We make the usual assumption of polyhedral focal elements Ω_i. The problem of interest here is the propagation of the uncertainty on x to the output of the function y.

In reliability problems the function g, typically called limit state function, expresses the difference between a design allowable (e.g. yield stress or maximum allowable take-off length) and the corresponding quantity of interest for the design (e.g. the stress in a structure as a function of the geometric parameters of the structure, the aircraft take-off length as a function of aircraft parameters such as engine thrust and wing area etc.). Thus when the value of the function g becomes negative it corresponds to the constraints being violated (possibly meaning failure).

A critical region (i.e. failure region) is then defined as:

$$Fail = \left\{ x \in \mathbb{R}^n / g(x) \le 0 \right\} \tag{6}$$

In the current work we seek to calculate the belief and plausibility of the critical region $Fail$, given the evidence-based input uncertainty on x and given the multidimensional nature of the function g:

$$Pl(Fail) = \sum_{i \in J^-} m_i \tag{7}$$

where $J^- = \left\{ 1 \le i \le k / \exists x \in \Omega_i / g(x) \le 0 \right\}$ (8)

$$Bel(Fail) = \sum_{i \in J^+} m_i \tag{9}$$

where $J^+ = \left\{ 1 \le i \le k / \forall x \in \Omega_i / g(x) \le 0 \right\}$ (10)

Note that $\exists$ means “there exists” and $\forall$ means “for any”, while k is the total number of focal elements.

The formulation of the failure region is typical of reliability problems, which have been considered before within the evidence theory framework [16][17]. Compared to earlier works the reliability formulation presented here has the particularity of considering the reliability of the entire system at once instead of considering reliability constraints at each component level (components of the vector output of g here). The usual formulations with reliability constraints for each component involve the implicit assumption that these components are independent. While this is often true or approximately verified there are also situations where the various components are not independent, in which case it is important to express the reliability of the entire system simultaneously in order to fully account for the interactions in the system.

Independence of the output components is considered here in the sense of non-interactivity (also known in evidence theory context as Shafer’s evidential independence [18]). Non-interactivity of the output components occurs for example when each component depends on input variables, which are not common among components. More precisely, if two output components had at least one common input variable they would become interactive, thus non-independent. In more mathematical terms independence can be expressed in terms of the property that the joint belief function can be entirely described by its marginals.

Note that in the case where p=1 (i.e. g is one-dimensional in the output space) we can write:

$$J^{-}_{\text{one-dimensional}} = \left\{ 1 \le i \le k \,/\, \underset{x\in\Omega_i}{Min}\left(g(x)\right) \le 0 \right\} \tag{11}$$

$$J^{+}_{\text{one-dimensional}} = \left\{ 1 \le i \le k \,/\, \underset{x\in\Omega_i}{Max}\left(g(x)\right) \le 0 \right\} \tag{12}$$

These are the usual expressions used for describing belief and plausibility for functions with one-dimensional output or when the outputs are considered to be independent and belief or plausibility are expressed for each output component. In order to reduce the computational cost of calculating belief and plausibility, by avoiding to solve a possibly global optimization problem, the vertex method [11],[12] has notably been proposed for one-dimensional functions. This approach works under the assumption that the function g is monotonic, in which case belief and plausibility functions can be easily calculated based only on the function evaluations at the vertices of each focal element (cf. [11]). An issue with the vertex method is that the number of vertices grows exponentially with the number of focal elements and thus also with the dimensionality of the input variable (i.e. with n). Furthermore the concept of monotonicity becomes problematic in cases where the function is multi-dimensional (i.e. p>1).

Sampling based methods for calculating belief and plausibility have also been suggested [19]-[21]. While these methods are always applicable, their numerical cost is largely problem dependent and computationally expensive problems (i.e. black box type finite element based simulations) may quickly become prohibitive to solve.

In cases where the function g is multidimensional in the output, which is the focus of the current paper, Eqs. 11 and 12 can no longer be directly applied. Indeed there is no total order in $\mathbb{R}^n$ so the optimization problems can no longer be formulated in the same way. We propose in the following sections various explicit formulations allowing to calculate the belief and plausibility functions in the multidimensional case (i.e. p>1) together with algorithms for their calculation.

B. Formulation for arbitrary functions

In this subsection we make no particular assumption, such as convexity or linearity, concerning the function g.

The formulation of the plausibility subset J^- of Eq. 8 can be expressed in terms of optimization problems as following:

$$J^- = \left\{ 1 \le i \le k \,/\, \underset{x \in \Omega_i}{Min} \left(\underset{1 \le j \le p}{Max} \left(g_j(x) \right) \right) \le 0 \right\} \tag{13}$$

where g $_j$() is the j-th component of the vector output of g. Note that the failure region is defined in terms of function g as shown in Eq. 6.

This formulation seeks to find at least an x^{*i} in Ω_i for which all the components of $g\left(x^{*i}\right)$ are non-positive, that is for a given Ω_i check if $\underset{x \in \Omega_i}{Min} \left(\underset{1 \le j \le p}{Max} \left(g_j(x) \right) \right) \le 0$. Note that for a given x^{*i} all the components of $g\left(x^{*i}\right)$ need to be non-positive which is expressed here as the maximum among the components being non-positive. Note that this formulation is not equivalent to checking $\underset{1 \le j \le p}{Max} \left(\underset{x \in \Omega_i}{Min} \left(g_j(x) \right) \right) \le 0$ since in the latter case the minimum obtained is not necessarily the same between the p different components of g.

The double optimization problem of Eq. 13 is not trivial to solve directly since the minimization over the index *j* precedes the optimization over the variable *x*, while the function g_j is a function of *x*. To overcome this difficulty we propose the following algorithm for calculating the plausibility subset J^- of Eq. 13:

Algorithm 1: Calculation of the plausibility subset J^- for arbitrary functions *g*.

$J^- == \varnothing$
for $i = 1, 2 \ldots k$ do
  Find $\left(x^{*i}, z^{*i}\right)$ solution of

$$\begin{cases} \text{minimize F} : (x, z) \mapsto z \\ \quad \text{subject to} : x \in \Omega_i \\ \qquad\qquad g_1(x) \le z \\ \qquad\qquad \vdots \\ \qquad\qquad g_p(x) \le z \end{cases}$$

  if $z^{*i} \le 0$ do
    $J^- == J^- \cup \{i\}$

The value of x^{*i} found in Algorithm 1 is the optimum argument of the problem $\underset{x \in \Omega_i}{Min}\left(\underset{1 \le j \le p}{Max}\left(g_j(x)\right)\right)$ while z^{*i} is the corresponding optimum, i.e. $z^{*i} = \underset{x \in \Omega_i}{Min}\left(\underset{1 \le j \le p}{Max}\left(g_j(x)\right)\right)$. Thus if $z^{*i} \le 0$ we found x^{*i} in Ω_i for which all the components of $g\left(x^{*i}\right)$ are non-positive, thus the index *i* corresponding to the current focal element Ω_i belongs to the plausibility subset J^-.

Similarly, the formulation of the belief subset J^+ of Eq. 10 can be expressed in terms of optimization problems as following:

$$J^+ = \left\{1 \le i \le k \,/\, \underset{1 \le j \le p}{Max}\left(\underset{x \in \Omega_i}{Max}\left(g_j(x)\right)\right) \le 0\right\} \tag{14}$$

To calculate the belief subset we need to check that for all *x* within Ω_i we have all the components of $g(x)$ that are non-positive, that is for a given Ω_i check if $\underset{1 \le j \le p}{Max}\left(\underset{x \in \Omega_i}{Max}\left(g_j(x)\right)\right) \le 0$.

Note that this formulation is here equivalent to checking $\underset{x\in\Omega_i}{Max}\left(\underset{1\le j\le p}{Max}\left(g_j(x)\right)\right)\le 0$, however the definition of Eq. 10 expressing the belief subset can be formulated in the form expressed in Eq. 14 since the optimum among the different components of *g* does not need to be the same. Indeed for the belief subset we need to check that all *x* in Ω_i satisfy $g(x)\le 0$. Thus, it is enough to check that for all the components of *g* (and thus for the maximum component) the maximum over *x* is non-positive. To solve this double optimization problem and express the belief subset J^+ we propose following algorithm:

Algorithm 2: Calculation of the belief subset J^+ for arbitrary functions Φ.

$J^+ == \varnothing$
for $i = 1,2...k$ do
  for $\mathrm{j} = 1,2...p$ do
    Find x^{*i} and $y_j^{*i} = g_j\left(x^{*i}\right)$ solution of
$$\begin{cases}\text{maximize } \mathrm{g}_j : x \mapsto y_j = g_j(x) \\ \qquad \text{subject to} : x\in\Omega_i\end{cases}$$
  if $\forall j \;\; y_j^{*i} \le 0$
    $J^+ == J^+ \cup \{i\}$

In the most general case the two optimization problems involved in algorithms 1 and 2 are global optimization problems. Often though additional properties can be known about the function *g* (e.g. convexity, linearity over each Ω_i). These properties can be either strictly verified or only approximately, for example by making simplifying assumptions such as approximate linearity of the functions over each focal element. In the next subsections we will derive the formulations and algorithms for calculating belief and plausibility functions that work for a multidimensional function *f* under specific assumptions such as convexity or linearity.

C. Formulation for convex functions

If the limit state function *g(x)* is convex over each focal element Ω_i, the calculation of the plausibility function in Algorithm 1 can be simplified and its numerical cost reduced, by using convex optimization techniques to solve the minimization problem, since both the objective function and the constraints in Algorithm 1 are then convex.

Similarly we can simplify the calculation of the belief function subset of Algorithm 2 under the assumption of a convex limit state function. Indeed, if g is convex we obtain a convex maximization problem. The maximum of a convex function will necessarily lie at one of the vertices of Ω_i, which drastically reduces computational cost.

D. Formulation for linear functions

If the function g is linear over each focal element Ω_i, the calculation of the belief and plausibility functions can be further simplified. The aim of the following algorithm is to be able to use linear programming methods, such as simplex based methods or interior points methods [22], which are computationally very efficient. To achieve that we need to rewrite the optimization problems in such a way as to make sure that both the objective functions and the constraints are linear over each focal element. Note that the assumption on the behavior of function g (i.e. its convexity or linearity) is an assumption of local behavior not global behavior. Indeed the methods that we propose require convexity or linearity only over the domain of each focal element (i.e., using the notations introduced in section II, over the domain of each Ω_i, see also Fig. 1). This is because the optimization problems need to be solved separately over each focal element's domain (cf. Eq. 13 and 14).

In the following section we make the assumption that over each focal element Ω_i the limit state function is linear, i.e. that the restriction g^i of g over the focal element Ω_i can be written as:

$$g^i(x) = A^i x + b^i \tag{15}$$

where A^i is an p x n matrix and b^i is a vector of size p x 1. Note that as before x is a vector of size n x 1.

Furthermore $x \in \Omega_i$ with Ω_i polyhedron is also written in linear algebra terms as:

$$B^i x \le c^i \tag{16}$$

where B^i is an $2n$ x n matrix and c^i is a vector of size $2n$ x 1. Note that the $2n$ number of lines since we have an upper and lower bound on the polyhedron.

In order to be able to apply linear programming optimization algorithms we need to make sure that the objective functions and constraints of the optimization problems of Algorithm 1 are linear in terms of the variable (x, z), and not just in terms of the variable x. To achieve this, the algorithm is rewritten under the following form:

Algorithm 3: Calculation of the plausibility subset J^- for the case g^i linear ($g^i(x) = A^i x + b^i$).

$J^- == \varnothing$
for $i = 1, 2 \ldots k$ do

Find by linear programming $\hat{x}^{*i} = \begin{pmatrix} x^{*i} \\ z^{*i} \end{pmatrix}$ solution of

$$\begin{cases} \text{minimize } \hat{F} : \hat{x} = \begin{pmatrix} x \\ z \end{pmatrix} \mapsto z \\ \quad \text{subject to : } \hat{A}^i \hat{x} \le -\hat{b}^i \\ \text{where } \hat{A}^i = \begin{pmatrix} & -1 \\ A^i & \vdots \\ & -1 \\ B^i & 0 \end{pmatrix} \text{ and } \hat{b}^i = \begin{pmatrix} -b^i \\ c^i \end{pmatrix} \end{cases}$$

if $z^{*i} \le 0$ do

$J^- == J^- \cup \{i\}$

The function $\hat{F}$ is indeed linear in terms of $\hat{x}$. Furthermore since $\hat{A}^i$ is a $(p+2n)$ x n+1 matrix, the constraints are indeed linear in terms of $\hat{x}$ as well. The basic idea behind this algorithm remains however the same as for Algorithm 1.

Similarly we use linear programming for calculating J^+ as shown in Algorithm 4 below:

Algorithm 4: Calculation of the belief subset J^+ for the case g linear.

$J^+ == \varnothing$
for $i = 1, 2 \ldots k$ do

for $\mathrm{j} = 1, 2 \ldots p$ do

Find by linear programming x^{*i} and $y_j^{*i} = \left(A^i x^{*i} + \mathrm{b}^i\right)_j$ solution of

$$\begin{cases} \text{maximize } \mathrm{g}_j^i : x \mapsto y_j^i = \left(A^i x + \mathrm{b}^i\right)_j \\ \qquad\qquad \text{subject to : } B^i x \le c^i \end{cases}$$

if $\forall j \;\; y_j^{*i} \le 0$

$J^+ == J^+ \cup \{i\}$

Algorithm 4 is a direct reuse of Algorithm 2 since the components y_j^i of the output vector are linear and the constraint is also linear because the focal elements are polyhedrons.

We can note that both for algorithms 3 and 4 the vertex method is applicable for solving the linear optimization problems. Such an approach can be a convenient alternative to linear programming algorithms in cases of a small dimensionality of the output (i.e. *m*). In case of algorithm 3 however, the vertex method needs to be applied to the specific objective functions and constraints and not just to the limit state function and focal element as in algorithm 4.

Finally note that while algorithms 3 and 4 can be applied as an exact method in cases where the function is indeed linear, they can also be applied as approximate methods by linearizing the function over each focal element under the assumption that the non-linearities of the function are small over each of the focal elements. While of course not always verified, this may be the case in real world problems if the uncertainties considered are relatively small. An application of the approximate method to preliminary aircraft design will be presented in section 5.

To close this section we provide below an overview of the computational complexity of the methods proposed. Let us denote by *Co(n,p)* the complexity of solving an optimization problem with *n* design variables and *p* constraints. Then the complexity of the proposed algorithm for calculating the plausibility is equal to *k*Co(n,p)*, while the complexity of the algorithm for calculating the belief is equal to *k*p*Co(n,1)*. Both algorithms have a complexity, which is linear in terms of the complexity of the optimization problem *Co(n,p)*. In general cases (arbitrary function *g*) the complexity of solving the optimization problem with *n* design variables and *p* constraints is exponential in *n* and thus may quickly become intractable. However, under regularity assumption, such as convexity or linearity, the complexity becomes polynomial in *n* [23]. For example for linear or convex quadratic functions the complexity of solving the optimization problem using interior point algorithms is $Co(n,p) = O(n^{3.5})$ [24].

IV. Analytical example problem

A. Problem description

The example problem presented in this section presents the uncertainty propagation through a multidimensional function. It aims particularly at illustrating the significance of treating at once the different components of the output function rather than assuming them to be independent. For this purpose we use here an analytical function having two input parameters and two output components. The function is defined as follows:

$$\begin{aligned} & f : \mathbb{R}^2 \to \mathbb{R}^2 \\ & \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \mapsto \begin{Bmatrix} y_1 = 2x_1 + x_2 \\ y_2 = 3x_1 - 4x_2 \end{Bmatrix} \end{aligned} \tag{17}$$

We consider a generic problem where a failure region is defined on $\{y_1, y_2\}$ and we seek the belief and plausibility of this failure region. The failure region is defined by:

$$Fail = \left\{ y \in \mathbb{R}^2 \middle/ 11 \le y_1 \le 18 \text{ and } -20 \le y_2 \le 3 \right\} \tag{18}$$

The uncertainty in the failure region stems from uncertainty on the two input parameters x_1 and x_2. We assume that this uncertainty is of epistemic nature and that it is explicited using evidence theory. Following BPA structure is assumed on the two input parameters.

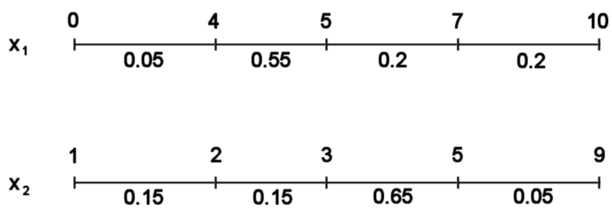

Figure 3. BPA uncertainty structure assumed on the input parameters. Upper numbers provide the bounds on the variables while lower numbers provide the corresponding BPA.

Note that the function *f* is linear and that the two output quantities are not independent and that we seek the belief and plausibility of the failure region at once, and not of each component independently . This requires treating the two components y_1 and y_2 at once in order to obtain exact results for belief and plausibility.

In order to use the same notations as in section III for the definition of the limit state function we introduce the following matrix *D* and vector *l* in order to write the limit state function as $g = Dy - l$.

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{19}$$

$$l = \begin{pmatrix} 18 \\ 3 \\ 11 \\ -20 \end{pmatrix} \tag{20}$$

B. Belief and plausibility results

In order to calculate the belief and plausibility of the failure zone we apply Algorithms 3 and 4 of section III.D. These were implemented using Matlab's linprog function, which utilizes a variation of the simplex method. The belief and plausibility values found are provided in Table 1.

Table 1. Belief and plausibility results for the analytical example problem

Quantity	Value
$Pl(Fail)$	0.6425
$Bel(Fail)$	0.3575

A graphical representation of the uncertainty structure in the output space as well as the representation of the failure region is provided in Figure 4. We can note that only one subset is fully included in the failure region and will thus contribute to the belief calculation. On the other hand seven subsets are fully or partially included in the failure region and thus contribute to the plausibility calculation.

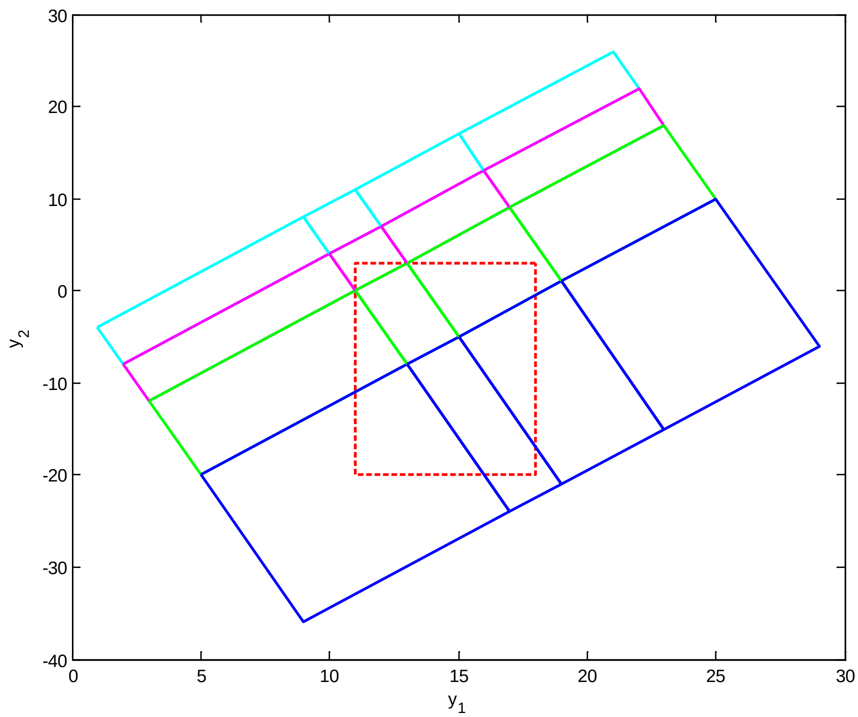

Figure 4. Graphical representation of the uncertainty structure in the output space (yellow, magenta, green and blue rectangles) together with the failure region (red dotted rectangle).

In order to show the error that would have been made by not considering the two output variables y_1 and y_2 at once, we also provide the results obtained by assuming independence (in the evidential sense [15],[18]) between y_1 and y_2 and using:

$$Pl_{indep}(Fail) = Pl(Fail(y_1)) \times Pl(Fail(y_2)) \tag{21}$$

$$Bel_{indep}(Fail) = Bel(Fail(y_1)) \times Bel(Fail(y_2)) \tag{22}$$

Where $Fail(y_1)$ is the failure region over y_1 only, independently of y_2, and similarly for $Fail(y_2)$.

Under these conditions we find the results provided in Table2:

Table 2. Belief and plausibility results under assumption of independence

Quantity	Value
$Pl_{indep}(Fail)$	0.7776
$Bel_{indep}(Fail)$	0.1628

These two values of belief and plausibility calculated under the approximation of independence are relatively far away from the true values of belief and plausibility calculated in Table 1. This is because the assumption of independence is far from being true as can be seen in Figure 4 where the uncertainty structure on y_1 and y_2 is significantly tilted compared to the horizontal and vertical axes.

The interest of the belief and plausibility methods for multidimensional functions that we propose lie accordingly mainly in situations where we have a function with multiple outputs that are not independent. An application to a preliminary aircraft design problem will be presented next.

V. Preliminary aircraft design application

A. Preliminary aircraft design problem

Preliminary sizing is the first step in aircraft design and serves to define some of the basic characteristics of the future airplane [25],[26]. At this stage the aircraft does not have any predefined shape or size but it is already possible to enumerate some of the basic characteristics sought, such as airplane configuration, wing aspect ratio, cruise speed, propulsion system. Based on these considerations the designer can formulate realistic performance constraints that the aircraft will have to meet, some being operational requirements (e.g. range), while others stemming from certification rules (e.g. take-off field length). Typically, requirements are considered on payload, range, approach, cruise and stall speed, take-off and landing field length as well as different climb gradients.

Preliminary sizing then involves determining following basic aircraft parameters: take-off mass, fuel mass, wing area and engine power (at take-off).

A critical step at the heart of preliminary sizing is the determination of the two parameters that are power to mass ratio *PM* and wing loading *WL*, defined in Eqs. 23 and 24.

$$PM = \frac{P_{TO}}{m_{MTO}} \tag{23}$$

$$WL = \frac{m_{MTO}}{S_W} \tag{24}$$

where P_{TO} is the take-off power, m_{MTO} is the maximum take-off mass and S_W is the wing area.

This determination is usually done using a matching chart as the one shown in Figure 5. The matching chart is a graphical representation of the optimization involved where the aim is to achieve the highest possible wing loading as well as the smallest possible power to mass ratio given the constraints enumerated earlier (operational and certification requirements). The corresponding design point fixes the values of *PM* and *WL*.

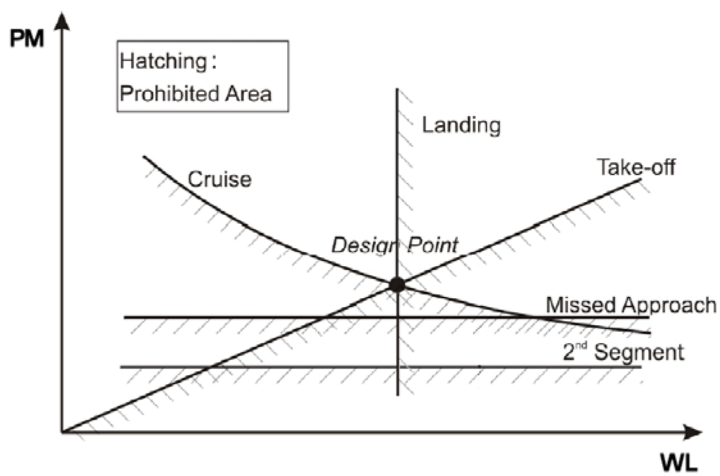

Figure 5. Example of matching chart for preliminary aircraft sizing, typical of a propeller airplane.

The chart in Figure 5 is typical of a propeller airplane, which is the aircraft type considered in this application problem. As often for preliminary sizing matching charts several constraints are inactive (e.g. missed approach climb gradient and 2nd segment climb gradient). The numerical values for the propeller airplane design that we considere here are taken from [27]. In this case the design point is determined by the intersection of three of the constraints (cf. [27]). For simplicity and since they define the same design point we will only consider the take-off and landing constraints for uncertainty propagation.

Once the design point found, the corresponding wing loading and power to mass ratio are fixed and are provided as inputs to the next design phase. However during the detailed aircraft design process it may be difficult to achieve the exact values of these parameters, due to various constraints (e.g. technical, financial). There is thus some uncertainty present about the exact values the parameters will take on the airplane coming out of the full design cycle. This uncertainty is of epistemic nature (lack of knowledge) and we model it here using evidence theory. The preliminary designer may then be interested in assessing the effects this uncertainty would have on the various aircraft characteristics and in particular those that involve active constraints (take-off and landing length here). He could then calculate the risk (in terms of belief and plausibility) that the various requirements are not met after the first design cycle.

We consider that we have epistemic uncertainty on the wing loading and on the power to mass ratio due to the early phase of the design process. We also consider we have epistemic uncertainty on the operating conditions requirements that will be imposed on the aircraft (e.g. air density ratio at which it can take-off etc). This is because while it is nice to be able to claim having an airplane that can operate in very extreme conditions, constraints during the subsequent design stages might impose settling for less extreme requirements.

These uncertainties will have an effect on the take-off length s_{TO} and the landing length s_L, which can be expressed as shown in Eqs. 25 and 26 below. For details on the derivation of these expressions we refer the reader to [27].

$$s_{TO} = \frac{WL \cdot k_{TO} \cdot V \cdot g}{PM \cdot \sigma \cdot C_{L,\max,TO} \cdot \eta_{P,TO}} = \alpha_{TO} \cdot \frac{WL}{PM \cdot \sigma} \tag{25}$$

where WL is the wing loading, PM the power to mass ratio and σ the air density ratio (actual air density at take-off over air density given by the International Standard Atmosphere). V is the average speed during take-off, g the gravity constant, $C_{L,max,TO}$ the maximum lift coefficient in take-off configuration, and $\eta_{P,TO}$ is the propeller efficiency at take-off. k_{TO} is a proportionality constant calculated from regression analysis over historical data of this type of aircraft [27].

$$s_L = \frac{WL \cdot \frac{m_{ML}}{m_{MTO}}}{k_L \cdot \sigma \cdot C_{L,\max,L}} = \alpha_L \cdot \frac{WL}{\sigma} \tag{26}$$

where m_{ML}/m_{MTO} is the ratio of the maximum mass at landing over the maximum mass at take-off, σ is the air density ratio and $C_{L,max,L}$ the maximum lift coefficient in landing configuration. k_L is a proportionality constant calculated from regression analysis over historical data of this type of aircraft [27].

All the parameters that are not varied during this study are aggregated into the two constants α_{TO} and α_L. These constants are calculated using the numerical values from [27], which gives α_{TO}=625 S.I. and α_L=2.83 S.I.

The assumed uncertainty structure on wing loading, power to mass ratio and density ratio is provided in Figure 6 in terms of the corresponding basic probability assignment (BPA).

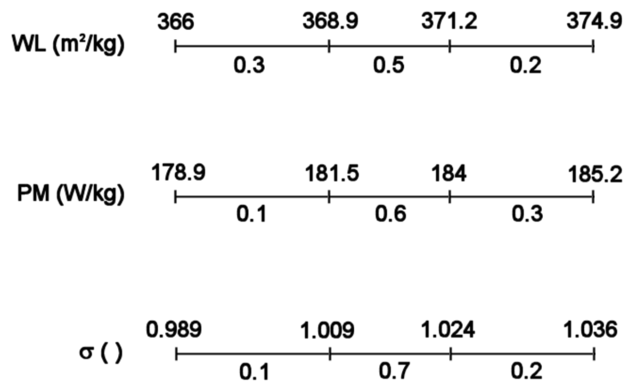

Figure 6. BPA uncertainty structure assumed on the input parameters. Upper numbers provide the bounds on the variables while lower numbers provide the corresponding BPA.

B. Belief and plausibility results

We seek here the belief and plausibility that, given the uncertainties defined above, the actual take-off and landing length remain within the imposed constraints. The take-off and landing field constraints are [27]:

$$NonFail = \left\{ \left(s_{TO}, s_L \right) \in \mathbb{R}^2 \middle/ s_{TO} \leq 1290m \text{ and } s_L \leq 1058m \right\} \tag{27}$$

In order to calculate the belief and plausibility of non-failure we use several of the algorithms proposed in section III. First, we can note that the function of (s_{TO}, s_L) is convex in terms of the uncertain input variables, thus it will be convex over each focal element on the input variables. Accordingly we applied the convex variation of

Algorithms 1 and 2 described in section III.C. in order to obtain the exact belief and plausibility values. The values found are given in Table 3.

Table 3. Belief and plausibility results for the preliminary aircraft design problem

Quantity	Value
$Pl(NonFail)$	1
$Bel(NonFail)$	0.928

The physical interpretation of the results is that, given the uncertainties considered, it is totally plausible (*Pl*=1) that the final airplane design will meet the take-off and landing length constraints. On the other hand, in the worst case, there is a chance of 0.928 (Bel=0.928) that the airplane design will meet the constraints and not require a redesign cycle.

At this point we also want to check how the linear approximation method for calculating belief and plausibility behaves on this example. Note that while there will be little computational cost saving by doing so on this example due to the low dimensionality we just wanted to test the accuracy of the linear approximation method. Accordingly we used algorithms 5 and 6 of section III.D by linearizing the function (s_{TO}, s_L) over each focal element (around the center of the focal element, using the jacobian matrix).

By doing so the same belief and plausibility values as the ones found with the exact method (Table 3) were obtained (up to three significant digits). This is mainly because the focal elements are relatively small meaning that the linear approximation will be quite accurate. Unless in presence of a function with large non-linearities over very small scales this is likely to be the case in many problems since the uncertainty remains in most cases small or moderate. Furthermore the failure region was not very close to any of the uncertainty structure subsets boundaries on the output variables, which is beneficial as well for the linear approximation method. This is also likely to be the case in other problems since it would be unlucky to have the constraints close to a large number of subsets simultaneously.

VI. Conclusion

Multidimensional function output requires more complex formulations and algorithms for calculating the belief and plausibility functions because there is no total order any more in $\mathbb{R}^n$. As a consequence, the algorithmic complexity can significantly increase in high dimensions. In this article we introduced one-dimensional optimization based algorithms designed specifically for evidence based uncertainty propagation through multi-dimensional functions, which thus allow avoiding the common approximation of independent outputs made when using one-dimensional uncertainty propagation algorithms on multi-dimensional functions.

We first proposed a general approach for uncertainty propagation through multidimensional functions, which does not rely on any specific assumptions on the nature of the multidimensional function. We then derived two algorithms for the cases where the functions are convex or linear over each focal element. These two approaches have the advantage of reducing the computational cost due to more efficient algorithms for convex and linear programming. Moreover the method based on linear functions is also applicable as an approximate method by linearizing an arbitrary function over each focal element.

The analytical illustrative example presented first showed the relevance of considering the multidimensional output all at once when the output components are non-independent. We found a non-negligible difference between the exact solution that considers non-independence and the approximate solutions under the assumption of independence. The second application concerned a preliminary aircraft sizing problem and illustrated both the exact calculation based on the convex variation of the proposed formulation as well as the approximate method based on linearization over each focal element. On this problem the linear approximation method led to almost identical results as the exact approach and this is likely to be the case in other problems where the non-linearities are small over each focal elements.

Our future work will be devoted to both aleatory and epistemic uncertainty propagation through multidimensional functions with non-independent output.

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